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What Graphs are 2-Dot Product Graphs?

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Abstract

From a set of d-dimensional vectors for some integer $d \geq 1$, we obtain a d-dot product graph by letting each vector $\mathbf{a}^u$ correspond to a vertex u and by adding an edge between two vertices u and v if and only if their dot product $\mathbf{a}^u \cdot \mathbf{a}^v \geq t$, for some fixed, positive threshold t. Dot product graphs can be used to model social networks. To understand the position of d-dot product graphs in the landscape of graph classes, we consider the case $d = 2$, and investigate how 2-dot product graphs relate to a number of other known graph classes.

Keywords: dot product graphs, social networks, graph classes

1   Introduction

Consider a social network in which each individual is friends with zero or more other individuals. In a vector model of the network, an individual u is described by a d-dimensional vector $\mathbf{a}^u$ for some integer $d \geq 1$ that expresses

the extent to which u has each of a set of d attributes (e.g. hobbies, opinions, music preferences, etc.). Two individuals are assumed to be friends if and only if their attributes are "sufficiently similar". There are many ways to measure similarity using a vector model (see, for example, [1,4,8,9,14]). In this paper, we use the *dot product model* that is, two individuals u and v are friends if and only if the dot product $\mathbf{a}^u \cdot \mathbf{a}^v \geq t$, for some fixed, positive threshold t. The corresponding graph G, in which each individual is a vertex and the friendship relation is described by the edge set, is called a *dot product graph* of *dimension* d or a *d-dot product graph*. We also say that the vector model $\{\mathbf{a}^u \mid u \in V\}$ with the threshold t is a *d-dot product representation* of G.

Dot product graphs have been studied from various perspectives. In particular, the study of dot product graphs as a model for social networks was initiated in a randomized setting [11,12,13,15,16], where the dot product of two vectors gives the probability that an edge occurs between the corresponding vertices. In a recent paper [5], we started the study of dot product graphs from an algorithmic perspective by considering the problems of finding a maximum independent set or a maximum clique in a d-dot product graph.

In this paper, we study dot product graphs from a graph-theoretic perspective. This line of research was initiated by Fiduccia et al. [3]. They showed that every graph on m edges has a dot product representation of dimension at most m [3], and conjectured that every graph on n vertices has a dot product representation of dimension at most $n/2$ (this conjecture has been confirmed, for example, for balanced complete bipartite graphs [3], chordal graphs [10], and graphs of girth at least 5 [10]). This led to the notion of the *dot product dimension* of a graph, which is the smallest d such that the graph has a d-dot product representation. Although graphs of dot product dimension 1 are easily understood and can be recognized in polynomial time (they are precisely the disjoint union of at most two threshold graphs [3]), we know comparatively little about graphs of dot product dimension 2 (or any higher fixed value).

This paper focusses on graphs of small dot product dimension, and in particular, on 2-dot product graphs. Kang and Müller [7] proved that recognizing graphs of any fixed dot product dimension $d \geq 2$ is NP-hard (membership in NP is still open). However, Fiduccia et al. [3] proved that every interval graph and every caterpillar is a 2-dot product graph; they also showed that not every tree is a 2-dot product graph (but trees have dot product dimension at most 3), that not every chordal graph is a 2-dot product graph (but chordal graphs have dot product dimension at most $\min\{\omega(G) + 1, n/2\}$ [3,10]), and that not all 2-dot product graphs are interval graphs (as the cycle on four vertices is a 2-dot product graph). Since there exist trees of dot product di-

mension 3, neither all outerplanar nor all planar graphs are 2-dot product graphs (however, Kang et al. [6] proved tight bounds of 3 and 4 respectively on the dot product dimension of these graphs). Li and Chang [10] showed that every wheel on six or more vertices has dot product dimension 3 (a wheel is a graph obtained from a cycle by adding a dominating vertex). Observe that in spite of these results, there are still many graph classes for which the relation to the class of graphs of small dot product dimension (and of dot product dimension 2 in particular) is unclear.

Our Results. We provide a more complete picture of the place of 2-dot product graphs in the landscape of known graph classes. We identify several new graph classes that are 2-dot product graphs, and show that certain graph classes are neither contained in the class of 2-dot product graphs nor do they contain all 2-dot product graphs. In particular, our work seems to provide evidence that no well-known graph class includes all 2-dot product graphs (however, we note explicitly here that we neither claim nor conjecture this).

2   Graph Classes and 2-Dot Product Graphs

Throughout, we assume that the threshold $t = 1$, unless stated otherwise. As any d-dot product graph has a representation with threshold 1 [3], this is no restriction. For space reasons, several proofs are omitted.

We start by observing that because the class of 2-dot product graphs is closed under vertex deletion, it can be characterized by a set of forbidden induced subgraphs. However, the class of 2-dot product graphs is not well-quasi-ordered that is, it has no *finite* set of forbidden induced subgraphs, because every wheel must be in this set of forbidden induced subgraphs. Indeed, a wheel minus a vertex is either a cycle or a fan, and thus has dot product dimension 2.

We note that 2-dot product graphs are not necessarily triangle-free, planar, nor H-minor-free for some fixed H, as they can contain arbitrarily large cliques. They are also not necessarily split, AT-free, even hole-free, or odd hole-free, because cycles of any length are 2-dot product (see [3] for cycles of length 4 or length at least 6; for the 5-vertex cycle this follows from Li and Chang's result [10] on graphs with girth at least 5). Also they are neither necessarily claw-free, as the claw has a 2-dot product representation (e.g. take $t = 3$ and vectors $(1, 1)$, $(1, 1)$, $(1, 1)$ and $(2, 2)$), nor circular-arc (e.g. take the complete graph on four vertices and add a pendant vertex to each vertex). Moreover, there exist 2-dot product graphs that are not a disk graph (take

the bi-4-wheel which can be represented as $(0, 5)$, $(\frac{1}{5}, 2)$, $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2})$, $(2, \frac{1}{5})$, $(5, 0)$ with $t = 1$). We note that grid graphs are not 2-dot product, as the 2×2 grid can easily be shown not to have a 2-dot product representation by following the proof for wheels in [10].

2.1 Co-Bipartite Graphs

We exhibit a sharp divide on whether co-bipartite graphs are 2-dot product graphs. First, we show that a complete graph minus a matching is still a 2-dot product graph.

Theorem 2.1 *Let G be a graph obtained from a complete graph by removing the edges of a matching. Then G has a 2-dot product representation.*

Proof. Let m be a positive integer. Let the vertex set of K_{2m} be denoted $\{v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_m\}$ and let I_m denote the set of edges $\{v_i w_i \mid 1 \leq i \leq m\}$ which form a perfect matching. We will first prove the special case of the theorem where $G = K_{2m} - I_m$. For a nonnegative integer k, let $b(k) = 2^k - 1$. For $1 \leq i \leq m$, let $\mathbf{a}^{v_i} = (1/b(i), b(i-1))$, $\mathbf{a}^{w_i} = (b(i-1), 1/b(i))$. We show this is a 2-dot product representation for $K_{2m} - I_m$. First consider pairs v_i, w_i:

$$\mathbf{a}^{v_i} \cdot \mathbf{a}^{w_i} = \frac{2b(i-1)}{b(i)} = \frac{2^i - 2}{2^i - 1} < 1.$$

We must show that all other pairs of distinct vertices have dot product at least 1. As each $b(k) \geq 1$, we have $\mathbf{a}^{v_i} \cdot \mathbf{a}^{v_j} \geq 1$ and $\mathbf{a}^{w_i} \cdot \mathbf{a}^{w_j} \geq 1$ for all i, j. Finally, for $i \neq j$,

$$\mathbf{a}^{v_i} \cdot \mathbf{a}^{w_j} = \frac{b(j-1)}{b(i)} + \frac{b(i-1)}{b(j)},$$

and one of the two quotients is at least 1, and the other is positive.

For the general case, choose the largest value of m such that $K_{2m} - I_m$ is an induced subgraph of G. Then every vertex not in this subgraph is adjacent to every vertex other than itself. We can obtain a 2-dot product representation of G using the representation described above for the vertices of $K_{2m} - I_m$ and by letting, for every other vertex u, $\mathbf{a}^u = (1, 1)$ (and by noting that every vertex has two positive coordinates one of which is at least 1). $\square$

For a positive integer $n \geq 3$, let A_{2n} be the (even) anti-cycle on $2n$ vertices. We note that even anti-cycles are co-bipartite and note the contrast between Theorems 2.1 and 2.2.
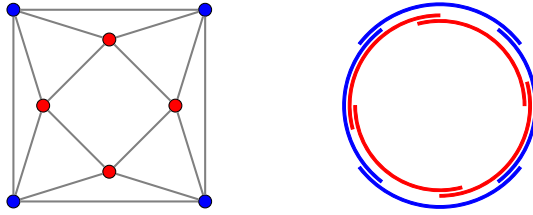
Fig. 1. The graph J and its representation as a unit circular-arc graph.

Theorem 2.2 *For $n \leq 3$, A_{2n} is a 2-dot product graph. For $n \geq 4$, A_{2n} is not a 2-dot product graph.*

2.2 Unit Circular-Arc Graphs and Split Graphs

Consider the unit sphere S^k. Then for some vector $\mathbf{c} \in S^k$, a *cap* of S^k is the set $\{\mathbf{x} \in S^k \mid \mathbf{c} \cdot \mathbf{x} \geq a\}$, where a is a real number in $(0, 1]$. We call the vector $\mathbf{c}$ the *center* of the cap, and $2 \arccos a$ its *angular diameter*. Observe that, given the range of a, the angular diameter of each cap lies in $[0, \pi)$. Fiduccia et al. [3] showed that a so-called capture graph of caps of S^k has dot product dimension at most $k + 1$, while Kang et al. [6] showed that a so-called contact graph of caps of S^k has dot product dimension at most $k + 2$. We consider *unit caps*: a set of caps of S^k is unit if all caps in the set have the same angular diameter $\theta \in [0, \pi/2)$.

Theorem 2.3 *The intersection graph of a set of unit caps of S^k has dot product dimension at most $k + 1$.*

It would seem that Theorem 2.3 also implies that all unit circular-arc graphs have dot product dimension at most 2. However, due to the limited angular diameter allowed in our definition of unit caps, this implication only holds if the graph has a unit circular-arc representation using unit caps of S^1. This is the case, for example, when the graph has no maximal independent set of size less than 4.

Theorem 2.4 *If G is a unit circular-arc graph with no maximal independent set of size less than 4, then G is a 2-dot product graph.*

Surprisingly, this restriction is not an artifact of our proof technique, but is actually needed: in Figure 1 is an example of a graph J that is a unit circular-arc graph and that has dot product dimension larger than 2. Note that such an example must have triangles, as every triangle-free unit circular-arc graph is either a path or a cycle and so has dot product dimension 2.
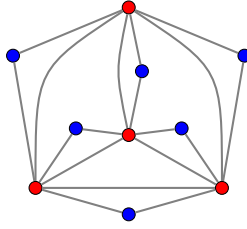
Fig. 2. The graph K. As the vertices can be partitioned into a clique and an independent set, it is a split graph.

Theorem 2.5 *There exist unit circular-arc graphs that do not have a 2-dot product representation.*

In Figure 2 is an example of a graph K that is a split graph and that has dot product dimension larger than 2. This give us our last result.

Theorem 2.6 *There exist split graphs that do not have a 2-dot product representation.*

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