Available online at www.sciencedirect.com
ScienceDirect
Electronic Notes in DISCRETE MATHEMATICS
www.elsevier.com/locate/endm

The Bruhat order on conjugation-invariant sets of involutions in the symmetric group

Mikael Hansson [1]

Department of Mathematics
Linköping University
Linköping, Sweden

Abstract

Let I_n be the set of involutions in the symmetric group S_n, and for $A \subseteq \{0, 1, \ldots, n\}$, let

$$F_n^A = \{\sigma \in I_n \mid \sigma \text{ has } a \text{ fixed points for some } a \in A\}.$$

We give a complete characterisation of the sets A for which F_n^A, with the order induced by the Bruhat order on S_n, is a graded poset. In particular, we prove that $F_n^{\{1\}}$ (i.e., the set of involutions with exactly one fixed point) is graded, which settles a conjecture of Hultman in the affirmative. When F_n^A is graded, we give its rank function. We also give a short new proof of the EL-shellability of $F_n^{\{0\}}$ (i.e., the set of fixed point-free involutions), which was recently proved by Can, Cherniavsky, and Twelbeck.

Keywords: Bruhat order, symmetric group, involution, conjugacy class, graded poset, EL-shellability

[1] Email: mikael.hansson@liu.se

1 Introduction

Partially ordered by the Bruhat order, the symmetric group S_n is a graded poset whose rank function is given by the number of inversions, and Edelman [4] proved that it is EL-shellable. Richardson and Springer [10] proved that the set I_n of involutions in S_n and the set F_n^0 of fixed point-free involutions are graded. Incitti [9] proved that the rank function of I_n can be expressed as the average of the number of inversions and the number of exceedances, and that I_n is EL-shellable. Hultman [8] studied (in a more general setting, which we shall describe shortly) F_n^0 and F_n^1, the set of involutions with exactly one fixed point. It follows that F_n^0 is graded and Hultman conjectured that the same is true for F_n^1. Can, Cherniavsky, and Twelbeck [3] recently proved that F_n^0 is EL-shellable.

We consider the following generalisation. For $a \in \{0, 1, \ldots, n\}$, let F_n^a be the conjugacy class in S_n consisting of the involutions with a fixed points, and for $A \subseteq \{0, 1, \ldots, n\}$, let

$$F_n^A = \bigcup_{a \in A} F_n^a.$$

Both I_n and F_n^A are regarded as posets with the order induced by the Bruhat order on S_n. Note that

$$F_n^A = \{\sigma \in I_n \mid \sigma \text{ has } a \text{ fixed points for some } a \in A\}.$$

Also note that for all elements in I_n, the number of fixed points is congruent to n modulo 2. Hence, we may assume that all members of A have the same parity as n.

Depicted in Figures 1 and 2, are the Hasse diagrams of I_4, F_4^0, and F_4^2.

Our main result is a complete characterisation of the sets A for which F_n^A is graded. In particular, we prove that F_n^1 is graded.

Informally, F_n^A is graded precisely when $A - \{n\}$ is empty or an "interval," which may consist of a single element if it is 0, 1, or $n - 2$. The following theorem, which is our main result, makes the above precise. It also gives the rank function of F_n^A when it exists.

Theorem 1 *The poset F_n^A is graded if and only if $A - \{n\} = \emptyset$ or $A - \{n\} = \{a_1, a_1 + 2, \ldots, a_2\}$ with $a_1 \in \{0, 1\}$, $a_2 = n - 2$, or $a_2 - a_1 \geq 2$. Furthermore, when F_n^A is graded, its rank function ρ is given by*

$$\rho(\sigma) = \frac{\operatorname{inv}(\sigma) + \operatorname{exc}(\sigma) - n + \tilde{a}}{2} + \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{otherwise,} \end{cases}$$

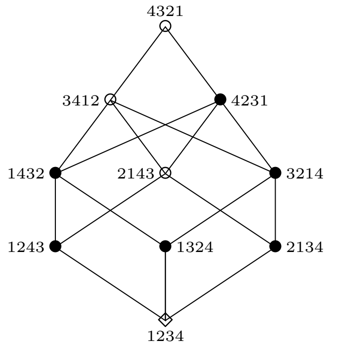

Figure 1. Hasse diagram of I_4 with the involutions with zero ($\circ$), two ($\bullet$), and four ($\diamond$) fixed points indicated.

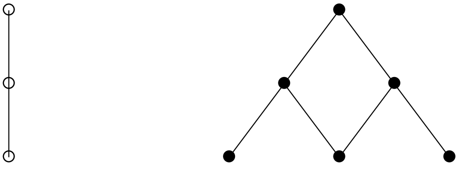

Figure 2. Hasse diagrams of F_4^0 (left) and F_4^2 (right).

where $\mathrm{inv}(\sigma)$ *and* $\mathrm{exc}(\sigma)$ *denote the number of inversions and exceedances, respectively, of* σ*, and* $\tilde{a} = \max(A - \{n\})$*. In particular,* F_n^A *has rank*

$$\rho(F_n^A) = \frac{n^2 - a^2 - 2n + 2\tilde{a}}{4} + \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{otherwise,} \end{cases}$$

where $a = \min A$*.*

The following result is direct consequence of Theorem 1.

Corollary 2 *The posets* F_n^0*,* F_n^1*,* F_n^{n-2}*, and* F_n^n *are the only graded conjugacy classes of involutions in* S_n*. Furthermore, the rank function* ρ *of* F_n^0 *and* F_n^1 *is given by*

$$\rho(\sigma) = \frac{\mathrm{inv}(\sigma) - \lfloor n/2 \rfloor}{2},$$

and the rank function ρ *of* F_n^{n-2} *is given by*

$$\rho(\sigma) = \frac{\mathrm{inv}(\sigma) - 1}{2}.$$

It is well known that F_n^{n-2} is graded (in fact, it coincides with the root poset of the Weyl group $A_{n-1} \cong S_n$). As was mentioned above, the gradedness of F_n^0 was proved by Richardson and Springer, and that of F_n^1 was conjectured by Hultman. These two posets are special cases of a more general construction from Hultman's paper [8], which we now briefly describe.

Given a finitely generated Coxeter system (W,S) and an involutive automorphism θ of (W,S) (i.e., a group automorphism θ of W such that $\theta(S)=S$ and $\theta^2=\mathrm{id}$), let

$$\iota(\theta)=\{\theta(w^{-1})w \mid w \in W\}$$

and

$$\mathfrak{I}(\theta)=\{w \in W \mid \theta(w)=w^{-1}\}$$

be the sets of *twisted identities* and *twisted involutions*, respectively. Clearly, $\iota(\theta) \subseteq \mathfrak{I}(\theta) \subseteq W$. Each subset of W is regarded as a poset with the order induced by the Bruhat order on W.

If W is finite, it contains a greatest element w_0, and $\theta(w)=w_0ww_0$ defines an involutive automorphism of (W,S). In this case, $\iota(\theta)$ is isomorphic to the dual of $[w_0]$, where $[w_0]$ is the conjugacy class of w_0, and $\mathfrak{I}(\theta)$ is isomorphic to the dual of $I(W)$, where $I(W)$ is the set of involutions in W. When W is the symmetric group S_n, $I(W)=I_n$, $[w_0]=F_n^0$ for n even, and $[w_0]=F_n^1$ for n odd.

Since $\iota(\theta)$ is graded whenever W is dihedral, as is easily seen, it follows from [8, Theorem 4.6 and Proposition 6.7] that $\iota(\theta)$ is graded whenever W is finite and irreducible, unless $W \cong S_{2n+1}$ with θ as above. It was conjectured by Hultman [8, Conjecture 6.1] that $\iota(\theta)$ is graded also in this last case. As we have seen, this is equivalent to F_n^1 being graded, which is the case (see Corollary 2). Hence, we get the following:

Theorem 3 *If W is finite, then $\iota(\theta)$ is graded.*

Let us also mention a connection to work by Richardson and Springer [10,11], who studied a partially ordered set V of orbits of certain symmetric varieties (depending on, inter alia, a group G). They did so by defining an order-preserving function $\varphi : V \to \mathfrak{I}(\theta) \subseteq W$ (where the Weyl group W depends on, inter alia, G).

To explain this connection, and for later purposes, define

$$F_n^{\leq a}=\bigcup_{i\geq 0} F_n^{a-2i} \quad \text{and} \quad F_n^{\geq a}=\bigcup_{i\geq 0} F_n^{a+2i},$$

and for $a_2 = a_1 + 2m$, where m is a positive integer, let

$$F_n^{a_1:a_2} = F_n^{\geq a_1} \cap F_n^{\leq a_2}.$$

Note that $F_n^{a_1:a_2}$ is not defined for $a_1 = a_2$.

It can be seen that $\mathfrak{I}(\theta)$, $\iota(\theta)$, and $F_n^{\geq a}$ for each $a \leq n-2$, are the images of such functions.

We also give a short new proof of the following result, which was recently proved by Can, Cherniavsky, and Twelbeck.

Theorem A ([3, Theorem 1]) *The poset F_n^0 is EL-shellable.*

2 A brief sketch of the proof of the main result

In this section, we state a number of lemmas and propositions, from which Theorem 1 easily follows.

We use several results due to Incitti. Here, we only state the one that we need in the proof of Theorem 1.

Lemma 4 ([9, Theorem 5.2]) *The poset I_n is graded with rank function ρ given by*

$$\rho(\sigma) = \frac{\mathrm{inv}(\sigma) + \mathrm{exc}(\sigma)}{2}.$$

The strategy for proving that a poset F_n^A is graded is as follows. We first prove that F_n^A has a maximum and that all its minimal elements have the same rank in I_n (see Propositions 6 and 7). We then prove that if $\sigma, \tau \in F_n^A$, then $\sigma \lhd \tau$ in F_n^A if and only if $\sigma \lhd \tau$ in I_n (one implication is obvious). This is done in Lemmas 9, 10, and 11. Since I_n is graded, it thus follows that F_n^A is graded.

In particular, when $F_n^A \in \{F_n^{\leq a}, F_n^{\geq a}\}$, to prove that $\sigma \lhd \tau$ in I_n if $\sigma \lhd \tau$ in F_n^A, we assume that $\sigma \ntriangleleft \tau$ in I_n, and consider the increasing and the decreasing σ-τ-chains in I_n. We then prove that either the element in the increasing chain that covers σ, or the element in the decreasing chain that is covered by τ, has to belong to F_n^A. This contradicts the fact that $\sigma \lhd \tau$ in F_n^A.

To prove that a poset F_n^A is not graded, we consider an interval $[\sigma, \tau]$, and then construct two σ-τ-chains in F_n^A of different lengths (see Propositions 13 and 14).

Let us first note the following fact:

Lemma 5 *For all n and all A, F_n^A is graded if and only if $F_n^{A-\{n\}}$ is graded.*

In the next two results, we describe the maximal and minimal elements of F_n^A.

Proposition 6 *For all n and all A, F_n^A has a $\hat{1}$. Furthermore,* $\mathrm{inv}(\hat{1}) = \frac{n-a}{2}(n+a-1)$ *and* $\mathrm{exc}(\hat{1}) = \frac{n-a}{2}$, *where* $a = \min A$.

Proposition 7 *For all n and all A, all minimal elements of F_n^A have rank $(n - \max A)/2$ in I_n.*

The following lemma will eventually allow us to conclude that $F_n^{\leq a}$, $F_n^{\geq a}$, and $F_n^{a_1:a_2}$ are graded.

Lemma 8 *If every cover in F_n^A is a cover in I_n, then F_n^A is graded.*

Proof This follows from Lemma 4 and Propositions 6 and 7. □

Lemma 9 *Let $\sigma \lhd \tau$ in $F_n^{\leq a}$. Then $\sigma \lhd \tau$ in I_n.*

Lemma 10 *Let $\sigma \lhd \tau$ in $F_n^{\geq a}$. Then $\sigma \lhd \tau$ in I_n.*

Lemma 11 *Let $\sigma \lhd \tau$ in $F_n^{a_1:a_2}$. Then $\sigma \lhd \tau$ in I_n.*

The proof of Lemma 10 requires more work than the proof of Lemma 9. The proof of Lemma 11 is largely a combination of the proofs of Lemmas 9 and 10.

Proposition 12 *The posets $F_n^{\leq a}$, $F_n^{\geq a}$, and $F_n^{a_1:a_2}$ are graded.*

Proof This follows from Lemmas 8, 9, 10, and 11. □

In the following two results, we describe the sets A for which F_n^A is not graded.

Proposition 13 *If there is an $i \in [2, n-4]$ such that $i \in A$ but $i-2, i+2 \notin A$, then F_n^A is not graded.*

The proof is similar to, but easier than, the proof of Proposition 14.

Proposition 14 *If there is an $i \notin A$ and a positive integer m such that $i-2, i+2m \in A - \{n\}$, then F_n^A is not graded.*

Figure 3 illustrates the proof when $n = 6$.

We are now ready to prove our main result:

Proof of Theorem 1 The first claim follows from Lemma 5 and Propositions 12, 13, and 14. (It is readily checked that if $F_n^{A-\{n\}}$ does not belong to $\{\emptyset, F_n^{\leq a}, F_n^{\geq a}, F_n^{a_1:a_2}\}$, then either there is an $i \in [2, n-4]$ such that $i \in A$ but

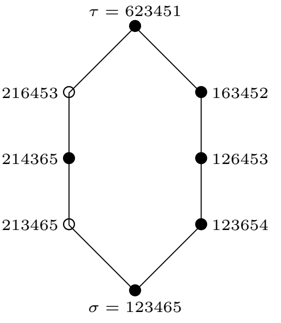

Figure 3. Two σ-τ-chains in I_6 of length 4, and two σ-τ-chains in $F_6^{\{0,4\}}$ of length 4 (right) and length 2 (left); the involutions marked by a $\bullet$ belong to $F_6^{\{0,4\}}$, and the involutions marked by a $\circ$ belong to $I_6 - F_6^{\{0,4\}}$.

$i-2, i+2 \notin A$, or there are an $i \notin A$ and a positive integer m such that $i-2, i+2m \in A - \{n\}$.) The second claim follows from Lemma 4, Proposition 7, and Lemmas 9, 10, and 11. The third claim follows from the second claim and Proposition 6. □

Acknowledgements

The author thanks Axel Hultman for helpful comments and fruitful discussions.

References

[1] Björner, A., *Shellable and Cohen-Macaulay partially ordered sets*, Trans. Amer. Math. Soc. **260** (1980), pp. 159–183.

[2] Björner, A. and F. Brenti, "Combinatorics of Coxeter groups," Graduate Texts in Mathematics **231**, Springer, New York, 2005.

[3] Can, M. B., Y. Cherniavsky and T. Twelbeck, *Lexicographic shellability of the Bruhat-Chevalley order on fixed-point-free involutions*, Israel J. Math., to appear .

[4] Edelman, P. H., *The Bruhat order of the symmetric group is lexicographically shellable*, Proc. Amer. Math. Soc. **82** (1981), pp. 355–358.

[5] Helgason, S., "Differential geometry, Lie groups, and symmetric spaces," Pure and Applied Mathematics **80**, Academic Press, New York, 1978.

[6] Hultman, A., *Fixed points of involutive automorphisms of the Bruhat order*, Adv. Math. **195** (2005), pp. 283–296.

[7] Hultman, A., *The combinatorics of twisted involutions in Coxeter groups*, Trans. Amer. Math. Soc. **359** (2007), pp. 2787–2798.

[8] Hultman, A., *Twisted identities in Coxeter groups*, J. Algebraic Combin. **28** (2008), pp. 313–332.

[9] Incitti, F., *The Bruhat order on the involutions of the symmetric group*, J. Algebraic Combin. **20** (2004), pp. 243–261.

[10] Richardson, R. W. and T. A. Springer, *The Bruhat order on symmetric varieties*, Geom. Dedicata **35** (1990), pp. 389–436.

[11] Richardson, R. W. and T. A. Springer, *Complements to: The Bruhat order on symmetric varieties*, Geom. Dedicata **49** (1994), pp. 231–238.

[12] Stanley, R. P., "Enumerative combinatorics. Vol. 1," Cambridge Studies in Advanced Mathematics **49**, Cambridge University Press, Cambridge, 1997.