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Third case of the Cyclic Coloring Conjecture

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Abstract

The Cyclic Coloring Conjecture of Ore and Plummer from 1969 asserts that the vertices of every plane graph with maximum face size Δ^* can be colored using at most $\lfloor 3\Delta^*/2 \rfloor$ colors in such a way that no face is incident with two vertices of the same color. The Cyclic Coloring Conjecture has been proven only for two values of Δ^*: the case $\Delta^* = 3$ is equivalent to the Four Color Theorem and the case $\Delta^* = 4$ is equivalent to Borodin's Six Color Theorem, which says that every graph that can be drawn in the plane with each edge crossed by at most one other edge is 6-colorable. We prove the case $\Delta^* = 6$ of the conjecture.

Keywords: Planar graphs, graph coloring, cyclic coloring, facial coloring.

Value of Δ^*	3	4	5	6	7	8	9	10
Upper bound	4	6	8	**9**	11	13	15	17
Source	[2,3]	[4,6]	[8]	**here**	[11]	[19]	[5]	[18]
Conjecture	4	6	7	9	10	12	13	15

Table 1
The known upper bounds for the Cyclic Coloring Conjecture.

1 Introduction

The Cyclic Coloring Conjecture of Ore and Plummer from 1969 [16] as a well-known graph coloring problem. It asserts that every plane graph with maximum face Δ^* has a cyclic coloring with at most $\lfloor 3\Delta^*/2 \rfloor$ colors, i.e. its vertices can be colored with at most $\lfloor 3\Delta^*/2 \rfloor$ colors in such a way that no two vertices on the same face get the same color. The case $\Delta^* = 3$ is equivalent to the Four Color Theorem. The only other known case is $\Delta^* = 4$, which is known as Borodin's Six Color Theorem [4,6]. This case of the conjecture is equivalent to the following statement: every graph embedded in the plane in such a way that each edge is crossed by at most one other edge is 6-colorable.

There has been a substantial amount of work on the conjecture; upper bounds for particular values of Δ^* are summarized in Table 1. The work on general bounds [5,16,8] culminated with currently the best known general bound $\lceil 5\Delta^*/3 \rceil$ due to Sanders and Zhao [18]. Amini, Esperet and van den Heuvel [1], proved that the conjecture holds asymptotically: for every $\varepsilon > 0$, there exists Δ_0 such that every plane graph with maximum face size $\Delta^* \geq \Delta_0$ has a cyclic coloring with at most $\left(\frac{3}{2} + \varepsilon\right) \Delta^*$ colors.

There has been no new exact results on the conjecture for more than 30 years. We resolve another case of the conjecture, proving the following.

Theorem 1.1 *Every plane graph with maximum face size at most six has a cyclic coloring using at most nine colors.*

The proof is based on a discharging argument with 103 discharging rules and 193 reducible configurations. Despite the high complexity of the argument,

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none of the proofs except for the proof of Lemma 4.3 are computer assisted.

We like to mention two related conjectures and refer to a survey [7] for more results. The conjecture of Plummer and Toft [17], studied e.g. in [9,12,13], asserts that every 3-connected plane graph with maximum face size Δ^* has a cyclic coloring using at most $\Delta^* + 2$ colors. The Facial Coloring Conjecture from [14], studied e.g. in [10,11,15], asserts that every plane graph has an ℓ-facial coloring with at most $3\ell+1$ colors, i.e. a coloring where vertices joined by a facial walk of length at most ℓ receive different colors. If true for a particular value of ℓ, then the Cyclic Coloring Conjecture holds for $\Delta^* = 2\ell + 1$.

2 Overview of the proof

We use the notation standard in the area of planar graph coloring. All graphs considered are plane graphs that can have parallel edges but not loops. A vertex of degree k is referred to as a k-vertex and a k-face is a face incident with k vertices. We use a $\leq k$-vertex, a $\geq k$-vertex, a $\leq k$-face and a $\geq k$-face in the obvious meanings. Two vertices are *facially adjacent* if they are incident with the same face and the *facial degree* of a vertex is the number of vertices facially adjacent to it. In a 2-connected plane graph, each face is bounded by a cycle and its proper connected subgraphs are referred to as *facial walks*.

We consider a *minimal counterexample* where the minimality is measured as the sum of the numbers of vertices and edges. A minimal counterexample must be 2-connected, have no parallel edges and have minimum facial degree at least 9. In addition, every cycle of length at most 6 must bound a face.

The existence of a minimal counterexample is excluded using the discharging method. We assign each k-vertex and each k-face $k-4$ units of charge. The total amount of the initial charges is -8. We then apply a set of discharging rules: some of the vertices and faces send charge to incident elements preserving the total sum of the charges. We show that a minimal counterexample cannot contain any of the specific 193 configurations, referred to as *reducible configurations*, and we argue that the final amount of charge of all elements is non-negative, which excludes the existence of a minimal counterexample.

3 Reducible configurations

The reducibility of most of the configurations is established as follows. We consider a minimal counterexample G containing the configuration, possibly add some edges and then contract one or more connected subgraphs to obtain a graph G' with maximum face size at most six. By the minimality of G, there exists a cyclic coloring of G' using at most nine colors. Most of the vertices of G keep the colors assigned, and the non-colored vertices are colored in a specific order. This order is chosen that each vertex is facially adjacent to vertices with at most eight different colors when it gets to be colored.

Our proof uses 186 reducible configurations with their reducibility established in the way that we have just described. There are seven additional configurations: four of them are reduced using list coloring techniques, two by a more involved variant of the argument given in the previous paragraph, and the reducibility of a 3-face incident with three 3-vertices by an ad hoc argument. This part of the proof is not computer assisted at all.

4 Analysis of final amount of charge

Even though our proof involves 103 discharging rules, the proofs of Lemmas 4.1 and 4.2 are not computer assisted. The analysis of the final amount of charge of $\geq$ 5-faces turned out to be too complex and was done with the assistance of a computer (a program is available as an ancillary file on arXiv).

Lemma 4.1 *The final amount of charge of every vertex of a minimal counterexample is non-negative.*

Lemma 4.2 *Every 3-face of a minimal counterexample receives at least one unit of charge.*

Lemma 4.3 *The difference between the amount of charge sent out and received by d-face, $d \in \{5, 6\}$, of a minimal counterexample is at most $d - 4$.*

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