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Decomposition of bi-colored square arrays into balanced diagonals

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Abstract

Given an $n \times n$ array M $(n \geq 7)$, where each cell is colored in one of two colors, we give a necessary and sufficient condition for the existence of a partition of M into n diagonals, each containing at least one cell of each color. As a consequence, it follows that if each color appears in at least $2n-1$ cells, then such a partition exists. The proof uses results on completion of partial Latin squares.

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1 Introduction

Let L be an $m \times n$ array with $m \leq n$. A *diagonal* in L is a subset of m cells of L such that no two cells are in the same row or in the same column. For a natural number k, such that $0 < k \leq n$, a *k-coloring* of L is an assignment of a color from a given set of k colors to each cell of L. Given a k-coloring of L, an l-transversal ($l \leq k$) is a diagonal of L in which at least l distinct colors are represented. A diagonal in a k-colored array L in which all k colors appear is called here *balanced*.

A known conjecture of Stein [11] asserts that for any n-coloring of an $n \times n$ array L, where each color appears in n cells, there exists an $(n-1)$-transversal. Stein's conjecture generalizes an earlier conjecture of Ryser and Brualdi [4], [9] which state that such a transversal exists for any n-coloring in which all colors in each row and each column are distinct.

A problem related to the Ryser-Brualdi-Stein Conjectures, is the search for conditions allowing a decomposition of a k-colored $m \times n$ array into disjoint m-transversals. For some conjectures and asymptotic results on the subject see [1], [2], [5], [6], [7].

In this paper we give a necessary and sufficient condition for a 2-colored $n \times n$ arrays to be partitioned into n disjoint balanced diagonals.

Definition 1.1 *We call a subset A of cells in an $n \times n$ array* improper *if there exists $i, j \in [n]$ such that each cell in A lies either in row i or in column j but not in both. Otherwise, a set is called* proper.

Figure 1 illustrates an improper set (marked with x's).

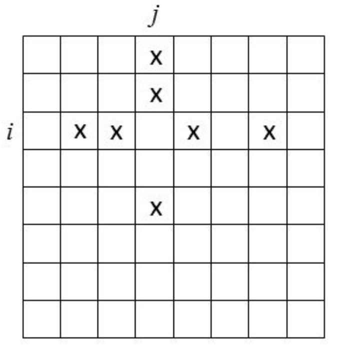

Fig. 1.

Our main result is the following theorem:

Theorem 1.2 *Suppose $n \geq 7$ and let L be an $n \times n$ array where each cell is* *colored red or blue. Then L can be partitioned into n balanced diagonals if and only if for each color there is a proper set of n cells colored with it.*

The proof of Theorem 1.2 relies upon results on completion of partial Latin squares.

2 Completion of partial Latin squares

A *Latin square* of *order* n is an $n \times n$ array filled with the symbols $1, \ldots, n$ so that all symbols in each row and each column are distinct. A diagonal in a Latin square consisting of equal symbols is called a *symbol diagonal.* A *partial Latin square* of *order* n and *size* k is an $n \times n$ array in which exactly k cells are filled. We shall use some results on completing a partial Latin squares to a Latin square:

Theorem 2.1 *(Smetaniuk [10]) A partial Latin square of order n and of size at most $n-1$ can be completed to a Latin square of order n.*

Theorem 2.2 *(Andersen and Hilton [3]) A partial Latin square of order n and of size n can be completed to a Latin square of order n, unless it can be brought by permuting rows and columns and possibly taking the transpose into one of the following two forms:*

- *Symbols $1, \ldots, x$ are in cells $(1,1), \ldots, (1,x)$ and symbols $x+1, \ldots, n$ are in cells $(2, x+1), \ldots, (n-x+1, x+1)$.*
- *Symbols $1, \ldots, x$ are in cells $(1,1), \ldots, (1,x)$ and the symbol $x+1$ is in cells $(2, x+1), (3, x+2) \ldots, (n-x+1, n)$.*

Observation 1 *Let L be an $n \times n$ array in which at least $n-1$ cells are colored blue. Then, there exists a partition of the cells of L into n disjoint diagonals, so that at least $n-1$ of them contain a blue cell.*

Proof. We assign the symbols $1, \ldots, n-1$ to the $n-1$ blue cells and obtain a partial Latin square. By Theorem 2.1, we can complete it to a Latin square in which the symbol diagonals form a partition of L into diagonals, so that at least $n-1$ of them contains a blue cell. □

Observation 2 *Let M be a colored $n \times n$ array containing a proper subset of n cells, which are all colored blue. Then, there is a partition of M into n diagonals, each containing a blue cell.*

Proof. Let B be the proper set of blue cells of size n. We assign the symbols $1, \ldots, n$ to the cells of B to obtain a partial Latin square L. Since B is proper and properness is preserved under permutation of rows and columns and taking the transpose, it follows from Theorem 2.2 that L can be completed to a Latin square. The symbol diagonals of this Latin square form a partition of M into diagonals, each containing a blue cell. □

It can be shown that $2n-2$ blue cells may not ensure the existence of a decomposition into diagonals, each containing a blue cell. But, since any set of $2n-1$ cells is proper, and thus contains a proper subset of size n, we have the following observation:

Observation 3 *Let M be a $n \times n$ array in which at least $2n-1$ cells are colored blue. Then, there is a partition of M into n diagonals, each containing a blue cell.*

3 Sketch of proof of the main result

For the proof of Theorem 1.2 we apply the following theorem of Ryser [8]:

Theorem 3.1 *Let $0 < r, s < n$ and let A be a partial Latin square of order n in which cell (i, j) in A is filled if and only if $i \leq r$ and $j \leq s$. Then A can be completed to a Latin square if and only if $N(i) \geq r + s - n$ for $i = 1, \ldots, n$, where $N(i)$ is the number of cells in A that are filled with i.*

Let L_b and L_r be the subsets of L consisting of blue and red cells, respectively. Without loss of generality we may assume that $|L_b| \leq |L_r|$. If $|L_b| < n$, then clearly there is no decomposition of L into balanced diagonals. Suppose $|L_b| \geq n$. If L_b does not contain a proper subset of size n, then L_b is improper. Suppose L_b is contained in row i and column j, then for any partition of L into diagonals, the diagonal through $L(i, j)$ will be contained in L_r. Thus, the condition is necessary.

In order to show that the condition is sufficient we assume, for contradiction, that a decomposition of L into balanced diagonals does not exist.

Here is a sketch of a proof:

(i) We show that if the contradiction assumption holds, then L_b contains two diagonals T_1 and T_2 such that $|T_1 \cap T_2| = 1$.

(ii) Suppose $T_1 \cap T_2 = \{c_{ij}\}$. Then, there exists a cell in $L_b \setminus (T_1 \cup T_2)$ which is not in row i and not in column j.

(iii) We show that L contains an $s \times t$ sub-rectangle R_1, such that $s + t = n$, $s - 1 \leq t \leq s + 1$ and $|R_1 \cap L_b| \geq n$.

The proof of this assertion utilizes the natural correspondence between

$n \times n$ arrays and the complete bipartite graph $K_{n,n}$ (where a diagonal corresponds to a matching).

(iv) We show that L contains a $p \times q$ sub-rectangle R, such that $p+q = n+1$, $p-2 \leq q \leq p+2$, $|R \cap L_b| \geq n$ and $|R \cap L_r| \geq n$.

In order to prove this assertion we note that, since $|L_b| \leq |L_r|$, the square L must contain an $s \times t$ sub-rectangle R_2 such that $|R_2 \cap L_r| \geq n$. If R_1 and R_2 coincide, we are done. Otherwise, we slide an $s \times t$ window, starting from R_1 to R_2 (Figure 2), so that in each step we either drop a row and add a row or drop a column and add a column. At some point we must move from a rectangle with at least n blue cells to a rectangle with at least n red cells (assuming $n \geq 7$). The union of these two rectangles is R.

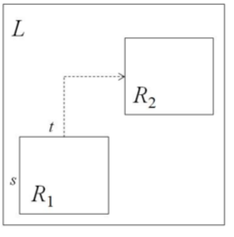

Fig. 2.

(v) We use Hall's theorem to show that we can fill n blue cells and n red cells of R with the numbers $1, \ldots, n$ so that each number appears once in a blue cell and once in a red cell, to form a partial Latin square.

(vi) We use Ryser's theorem 3.1 to show that R can be completed to a Latin square, in which all the symbol diagonal are balanced.

Since any set of $2n-1$ cells contains a proper subset of size n we have the following corollary:

Corollary 3.2 *Let L be a 2-colored $n \times n$ array with $n \geq 7$. If each color appears in at least $2n-1$ cells, then L can be partitioned into n balanced diagonals.*

The results in this paper originated from questions on edge colorings of the complete bipartite graph $K_{n,n}$. Thus, we formulate Corollary 3.2 in these terms.

Definition 3.3 *Let $f : E(K_{n,n}) \to \{1,2\}$ be a coloring. A matching in $M \subset E(K_{n,n})$ is called* balanced *if $f^{-1}(i) \neq \emptyset$ for $i = 1, 2$.*

Theorem 3.4 *Let $n \geq 7$ and let $f : E(K_{n,n}) \to \{1,2\}$ be a coloring. If $f^{-1}(i) \geq 2n-1$ for $i = 1,2$, then there exists a partition of $E(K_{n,n})$ into n disjoint balanced matchings.*

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