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Minimum degrees and codegrees of minimal Ramsey 3-uniform hypergraphs

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Abstract

A uniform hypergraph H is called k-Ramsey for a hypergraph F, if no matter how one colors the edges of H with k colors, there is always a monochromatic copy of F. We say that H is minimal k-Ramsey for F, if H is k-Ramsey for F but every proper subhypergraph of H is not. Burr, Erdős and Lovasz [S. A. Burr, P. Erdős, and L. Lovász, *On graphs of Ramsey type*, Ars Combinatoria 1 (1976), no. 1, 167–190] studied various parameters of minimal Ramsey graphs. In this paper we initiate the study of minimum degrees and codegrees of minimal Ramsey 3-uniform hypergraphs. We show that the smallest minimum vertex degree over all minimal k-Ramsey 3-uniform hypergraphs for $K_t^{(3)}$ is exponential in some polynomial in k and t. We also study the smallest possible minimum codegrees over minimal 2-Ramsey 3-uniform hypergraphs.

Keywords: minimal Ramsey hypergraph, minimum degree and codegree

1 Introduction and New Results

A graph G is said to be Ramsey for a graph F if no matter how one colors the edges of G with two colors, say red and blue, there is a monochromatic copy of F. A classical result of Ramsey [11] states that for every F there is an integer n such that K_n is Ramsey for F. If G is Ramsey for F but every proper subgraph of G is not Ramsey for F, then we say that G is minimal Ramsey for F. We denote by $\mathcal{M}_k(F)$ the set of minimal graphs G with the property that no matter how one colors the edges of G with k colors, there is a monochromatic copy of F in it, and refer to these as minimal k-Ramsey graphs for F. There are many challenging open questions concerning the study of various parameters of minimal k-Ramsey graphs for various F. The most studied ones are the classical (vertex) Ramsey numbers $r_k(F) := \min_{G\in\mathcal{M}_k(F)} v(G)$ and the size Ramsey number $\hat{r}_k(F) := \min_{G\in\mathcal{M}_k(F)} e(G)$, where $v(G)$ is the number of vertices in G and $e(G)$ is its number of edges. To determine the classical Ramsey number $r_2(K_t)$ is a notoriously difficult problem and essentially the best known bounds are $2^{(1+o(1))t/2}$ and $2^{(2+o(1))t}$ due to Spencer [13] and Conlon [4].

Burr, Erdős and Lovász [1] were the first to study other possible parameters of the class $\mathcal{M}_2(K_t)$. In particular they determined the minimum degree $s_2(K_t) := \min_{G\in\mathcal{M}_2(K_t)} \delta(G) = (t-1)^2$ which looks surprising given the exponential bound on the minimum degree of K_n with $n = r_2(K_t)$ (it is not difficult to see that $K_n \in \mathcal{M}_2(K_t)$). Extending their results, Fox, Grinshpun, Liebenau, Person and Szabó [7] studied the minimum degree $s_k(K_t) := \min_{G\in\mathcal{M}_k(K_t)} \delta(G)$ for more colors showing a general bound on $s_k(K_t) \leq 8(t-1)^6k^3$ and proving quasiquadratic bounds in k on $s_k(K_t)$ for fixed t. Further results concerning minimal Ramsey graphs were studied in [2,8,9,12,14].

Here we initiate the study of minimal Ramsey 3-uniform hypergraphs and provide first bounds on various notions of minimum degrees for minimal Ramsey hypergraphs. Generally, an r-uniform hypergraph H is a tuple (V, E) with vertex set V and $E \subseteq \binom{V}{r}$ being its edge set. Ramsey's theorem holds for r-uniform hypergraphs as well, as shown originally by Ramsey himself [11], and we write $G \longrightarrow (F)_k$ if G is k-Ramsey for F, i.e. if no matter how one colors the edges of the r-uniform hypergraph G, there is a monochromatic copy of F. We denote by $K_t^{(r)}$ the complete r-uniform hypergraph with t vertices, i.e.

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$K_t^{(r)} = ([t], \binom{[t]}{r})$, and by the hypergraph Ramsey number $r_k(F)$ the smallest n such that $K_n^{(r)} \longrightarrow (F)_k$. While in the graph case the known bounds on $r_2(K_t)$ are only polynomially far apart, already in the case of 3-uniform hypergraphs the bounds on $r_2(K_t^{(3)})$ differ in one exponent: $2^{c_1 t^2} \leq r_2(K_t^{(3)}) \leq 2^{2^{c_2 t}}$ for some absolute positive constants c_1 and c_2, and a similar situation occurs for higher uniformities. For further information on Ramsey numbers we refer the reader to the standard book on Ramsey theory [10] and for newer results to the survey of Conlon, Fox and Sudakov [5].

Given $\ell \in [r-1]$, we define the degree $\deg(S)$ of an ℓ-set S in an r-uniform hypergraph $H = (V, E)$ as the number of edges that contain S and we define the minimum ℓ-degree $\delta_\ell(H) := \min_{S \in \binom{V}{\ell}} \deg(S)$. For two vertices u and v we simply write $\deg(u, v)$ for the *codegree* $\deg(\{u, v\})$. Similar to the graph case we extend verbatim the notion of minimal Ramsey graphs to minimal Ramsey r-uniform hypergraphs in a natural way. That is, $\mathcal{M}_k(F)$ is the set of all minimal k-Ramsey r-uniform hypergraphs H, i.e. consisting of those with $H \longrightarrow (F)_k$ but $H' \not\longrightarrow (F)_k$ for all $H' \subsetneq H$. We define

$$s_{k,\ell}(K_t^{(r)}) := \min_{H \in \mathcal{M}_k(K_t^{(r)})} \delta_\ell(H), \tag{1}$$

which extends the introduced graph parameter $s_k(K_t)$. It will be shown actually that $s_{2,2}(K_t^{(3)})$ is zero and thus it makes sense to ask for the second smallest value of the codegrees. This motivates the following parameter $s'_{k,\ell}(K_t^{(r)})$:

$$s'_{k,\ell}(K_t^{(r)}) := \min_{H \in \mathcal{M}_k(K_t^{(r)})} \left(\min \left\{ \deg_H(S) \colon S \in \binom{V(H)}{\ell}, \deg_H(S) > 0 \right\} \right).$$

We prove the following results on the minimum degree and codegree of minimal Ramsey 3-uniform hypergraphs for cliques $K_t^{(3)}$.

Theorem 1.1 *The following holds for all $t \geq 4$ and $k \geq 2$*

$$\hat{r}_k(K_{t-1}) \leq s_{k,1}(K_t^{(3)}) \leq k^{20kt^4}. \tag{2}$$

The lower bound $\hat{r}_k(K_{t-1})$ is the *size-Ramsey number* for K_{t-1} and it was shown by Erdős, Faudree, Rousseau and Schelp [6] that $\hat{r}_k(K_\ell) = \binom{r_k(K_\ell)}{2}$. Using the lower bound on $r_k(K_\ell) \geq 2^{\frac{1-o(1)}{4} k\ell}$ (see e.g. [5]) we obtain $s_{k,1}(K_t^{(3)}) \geq 2^{\frac{1}{2}kt(1-o(1))}$.

Theorem 1.2 *Let $t \geq 4$ be an integer. Then,*

$$s_{2,2}(K_t^{(3)}) = 0 \text{ and } s'_{2,2}(K_t^{(3)}) = (t-2)^2.$$

Notice that with $s'_{2,2}$ we ask for the smallest *positive* codegree, while for $s_{2,2}$ we also allow the codegree to be zero. This in particular means that in *any* minimal 2-Ramsey hypergraph H for $K_t^{(3)}$ we have that a pair of vertices u and v are either not contained in a common edge or have codegree at least $(t-2)^2$. This might look surprising at the first sight since taking $K_n^{(3)}$ with $n = r_2(K_t^{(3)})$ and then deleting all edges that contain two distinguished vertices gives a non-Ramsey hypergraph.

2 Main tools

2.1 BEL-gadgets

We refer in the following to a coloring without a monochromatic copy of F as an *F-free coloring*. Our first tool is a result that asserts existence of non-k-Ramsey hypergraphs $\mathcal{H}$ for $K_t^{(3)}$ that impose certain structure on *all* $K_t^{(3)}$-free colorings of $E(\mathcal{H})$.

Theorem 2.1 *Let $k \geq 2$ and $t \geq 4$ be integers. Let H be a 3-uniform hypergraph with $H \not\to \left(K_t^{(3)}\right)_k$ and let $c\colon E(H) \to [k]$ be a k-coloring which avoids monochromatic copies of $K_t^{(3)}$. Then, there exists a 3-uniform hypergraph $\mathcal{H}$ with the following properties:*

(i) $\mathcal{H} \not\to \left(K_t^{(3)}\right)_k$,

(ii) *$\mathcal{H}$ contains H as an induced subhypergraph, i.e. $\mathcal{H}[V(H)] = H$,*

(iii) *for every coloring $\varphi\colon E(\mathcal{H}) \to [k]$ without a monochromatic copy of $K_t^{(3)}$, the coloring of H under φ agrees with the coloring c, up to a permutation of the k colors,*

(iv) *if there are two vertices $a, b \in V(H)$ with $\deg_H(a,b) = 0$ then $\deg_{\mathcal{H}}(a,b) = 0$ as well,*

(v) *if $|V(H)| \geq 4$ then for every vertex $x \in V(\mathcal{H}) \setminus V(H)$ there exists a vertex $y \in V(H)$ such that $\deg_{\mathcal{H}}(x,y) = 0$.*

This theorem is crucial for our constructions of minimal k-Ramsey hypergraphs and thus for giving upper bounds on $s_{k,1}(K_t^{(3)})$, $s_{k,2}(K_t^{(3)})$ and $s'_{k,2}(K_t^{(3)})$, respectively. For its proof we first show the existence of a 3-uniform hypergraph $\mathcal{H}$ and two edges $f, e \in E(\mathcal{H})$ with $|f \cap e| = 2$ and $e(\mathcal{H}[e \cup f]) = 2$ so that $\mathcal{H}$ is not k-Ramsey for $K_t^{(3)}$ with the property that any F-free k-coloring of $E(\mathcal{H})$ colors the edges e and f differently. Putting several copies of these hypergraphs together in an appropriate way we receive a hypergraph promised by Theorem 2.1 (and we refer to such $\mathcal{H}$ as BEL-gadgets). In the graph case similar gadgets (called *positive/negative signal senders*) were given first by Burr, Erdős and Lovász [1] in the case of two colors, and later generalized by Burr, Nešetřil and Rödl [2] and by Rödl and Siggers [12].

2.2 Random hypergraphs

The random hypergraph $H^{(3)}(n,p)$ is the probability space of all labeled 3-uniform hypergraphs on the vertex set $[n]$ with each edge appearing with probability p independently of all other edges. The following lemma is crucial for the upper bound (2) in Theorem 1.1.

Lemma 2.2 *Let $t \geq 4$ and $k \geq 2$ be integers. There is a 3-uniform hypergraph H on $n = k^{10kt^4}$ vertices, which can be written as an edge-disjoint union of k 3-uniform hypergraphs $H_1, \ldots, H_k$ with the following properties:*

(i) *for every $i \in [k]$, H_i contains no copies of $K_t^{(3)}$, and*

(ii) *for any coloring c of the edges of the complete graph K_n with k colors there exists a color $x \in [k]$ and k sets $S_1, \ldots, S_k$ that induce copies of K_{t-1} in color x under the coloring c such that $H_1[S_1] \cong \ldots \cong H_k[S_k] \cong K_{t-1}^{(3)}$.*

The rough idea of the proof of Lemma 2.2 is to take k random hypergraphs $H_1', \ldots, H_k' \sim H^{(3)}(n,p)$, with p being chosen appropriately. And then to show that, with positive probability, even after deleting those edges which appear in at least two hypergraphs H_i' or in a copy of $K_t^{(3)}$ inside some H_i', we are left with k edge-disjoint hypergraphs $H_1, \ldots, H_k$ that satisfy the conditions above.

For the details we refer the reader to the full version of our paper [3].

3 Proof of Theorem 1.1

Lower bound

Take a minimal k-Ramsey hypergraph $\mathcal{H}$ for $K_t^{(3)}$ together with a vertex $v \in V(\mathcal{H})$ such that $\deg(v) = \delta(\mathcal{H}) = s_{k,1}(K_t^{(3)})$. We know that there exists a $K_t^{(3)}$-free k-coloring of $\mathcal{H} \setminus \{v\}$ which cannot be extended to a $K_t^{(3)}$-free k-coloring of $\mathcal{H}$. But this implies that $\mathrm{link}_{\mathcal{H}}(v) \longrightarrow (K_{t-1})_k$ holds, where

$\text{link}_{\mathcal{H}}(v)$ is the link of v, i.e., the graph consiting of all edges e such that $e \cup \{v\} \in E(\mathcal{H})$. Therefore: $s_{k,1}(K_t^{(3)}) = \deg(v) \geq \hat{r}_k(K_{t-1})$.

Upper bound

The proof of our upper bound on $s_{k,1}(K_t^{(3)})$ makes use of the BEL-gadgets. We fix a 3-uniform hypergraph H as asserted by Lemma 2.2 and a $K_t^{(3)}$-free k-coloring c of $E(H)$ which colors each of the subhypergraphs H_i monochromatically with color $i \in [k]$. Applying Theorem 2.1 for this choice of H and c, we obtain a new hypergraph $\mathcal{H}'$, that contains H as an induced subhypergraph, and we extend it further to a hypergraph $\mathcal{H}$ by adding one new vertex v with the edges $\{v, a, b\}$ for all $\{a, b\} \in \binom{V(H)}{2}$, i.e. the link of v is $\text{link}_{\mathcal{H}}(v) := \binom{V(H)}{2}$. So, $\deg_{\mathcal{H}}(v) = \binom{n}{2} < k^{20kt^4}$ holds. Owing to the assertions on $\mathcal{H}'$ we have $\mathcal{H}' \not\longrightarrow (K_t^{(3)})_k$. On the other hand one can show $\mathcal{H} \longrightarrow (K_t^{(3)})_k$, which follows from Property (ii) of Lemma 2.2. Thus, we conclude that there needs to exist a minimal k-Ramsey hypergraph $\mathcal{H}''$ of $K_t^{(3)}$ with $\mathcal{H}' \subseteq \mathcal{H}'' \subseteq \mathcal{H}$ and $0 < \deg_{\mathcal{H}''}(v) < k^{20kt^4}$. $\square$

4 Proof of Theorem 1.2

The size of $s'_{2,2}$

For the proof of $s'_{2,2}(K_t^{(3)}) \geq (t-2)^2$ we take a minimal 2-Ramsey hypergraph H for $K_t^{(3)}$ together with two vertices u and $v \in V(H)$ such that $\deg_H(u, v) > 0$. We aim to show that $\deg_H(u, v) \geq (t-2)^2$, and thus, for contradiction, we assume the opposite. We then delete all edges containing both u and v in order to obtain a hypergraph H', which satisfies $H' \not\to \left(K_t^{(3)}\right)_2$. That is, we find a red-blue coloring c of $E(H')$ which does not create a monochromatic copy of $K_t^{(3)}$. Now, let $N(u, v) := \{w \in V(H) : \ \{u, v, w\} \in E(H)\}$, $\deg_H(u, v) = |N(u, v)|$, and fix a longest sequence $B_1, \dots, B_k$ of vertex disjoint sets of size $t-2$ in $N(u, v)$, such that both $B_i \cup \{u\}$ and $B_i \cup \{v\}$ span only blue edges under the coloring c in H'. By assumption on the codegree $\deg_H(u, v)$, we know that $k < t-2$. We then extend the coloring c to a coloring of $E(H)$ as follows. For each edge $e = \{u, v, w\} \in E(H)$ with $w \in \bigcup B_i$ we set $c(e) = \text{red}$, while for all other edges $e = \{u, v, w\} \in E(H)$ we set $c(e) = \text{blue}$. It then follows that under this coloring there is no monochromatic copy of $K_t^{(3)}$ in H, contradicting $H \to (K_t^{(3)})_2$.

For the proof of $s'_{2,2}(K_t^{(3)}) \leq (t-2)^2$ we first provide a hypergraph H as follows. We choose $V(H) := [(t-2)^2] \cup \{a, b\}$ together with a partition of $[(t-2)^2]$ into $(t-2)$ equal-sized sets $V_1,\ldots,V_{t-2}$. Moreover, we define $E(H)$ by taking all edges of the clique $K^{(3)}_{(t-2)^2+2}$ on the vertex set $\bigcup V_i \cup \{a,b\}$ and then deleting all edges that contain both a and b plus deleting all edges that cross exactly two different V_is and contain neither a nor b. For this particular hypergraph, we then define a red-blue-coloring c as follows: the edges contained in $V_i \cup \{a\}$ and in $V_i \cup \{b\}$ for $i \in [t-2]$ are colored *blue*, while the other edges of H are colored *red*. By construction of H this coloring does not produce a monochromatic copy of $K_t^{(3)}$.

Now, applying Theorem 2.1 to H and c, we obtain a 3-uniform hypergraph $\mathcal{H}$ which contains H as an induced subhypergraph such that $\mathcal{H} \not\rightarrow K_t^{(3)}$, $\deg_{\mathcal{H}}(a,b) = 0$ and such that any $K_t^{(3)}$-free red-blue coloring ϕ of $E(\mathcal{H})$ agrees on $E(H)$ with the coloring c up to permutation of the two colors. Extending this construction by adding to $\mathcal{H}$ all $(t-2)^2$ edges $\{a,b,u\}$ where $u \in [(t-2)^2]$, we finally end up in a hypergraph $\mathcal{H}'$ for which it is not difficult to see that $\mathcal{H}' \longrightarrow (K_t^{(3)})_2$. Thus, as $\mathcal{H} \not\rightarrow (K_t^{(3)})_2$, there needs to exist a minimal 2-Ramsey hypergraph $\mathcal{H}''$ of $K_t^{(3)}$ with $\mathcal{H} \subseteq \mathcal{H}'' \subseteq \mathcal{H}'$ and such that $0 < \deg_{\mathcal{H}''}(a,b) \leq (t-2)^2$, i.e., $s'_{2,2}(K_t^{(3)}) \leq (t-2)^2$.

Showing $s_{2,2}(K_t^{(3)}) = 0$.

Let us consider the previous construction of $\mathcal{H}'$ again. As $s'_{2,2}(K_t^{(3)}) = (t-2)^2$ was proven, we know that *any* minimal 2-Ramsey subhypergraph of $\mathcal{H}'$ for $K_t^{(3)}$ has to contain *all* $(t-2)^2$ edges that contain a and b, and in particular, any such minimal hypergraph $\mathcal{H}''$ needs to contain all vertices of the induced subhypergraph H. However, $\mathcal{H}''[V(H)] \not\longrightarrow (K_t^{(3)})_2$ holds, as can be seen by considering a red-blue-edge-coloring chosen uniformly at random and showing that the expected number of monochromatic copies of $K_t^{(3)}$ in $\mathcal{H}''[V(H)]$ is less than 1.

Thus, any minimal 2-Ramsey subhypergraph $\mathcal{H}''$ of $\mathcal{H}'$ has to contain at least one further vertex $x \notin V(H)$. Then, since $|V(H)| = (t-2)^2 + 2 \geq 6$, it follows by Property (v) of Theorem 2.1 that there exists a vertex $y \in V(H)$ such that $0 = \deg_{\mathcal{H}'}(x,y) \geq \deg_{\mathcal{H}''}(x,y)$, i.e., $s_{2,2}(K_t^{(3)}) = 0$. □

For the details we refer the reader to the full version of our paper [3].

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