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4-colorability of P_6-free graphs ⋆

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Abstract

In this paper we will study the complexity of 4-colorability in subclasses of P_6-free graphs. The well known k-colorability problem is NP-complete. It has been shown that if k-colorability is solvable in polynomial time for an induced H-free graph, then every component of H is a path. Recently, Huang [11] has shown several improved complexity results on k-coloring P_t-free graphs, where P_t is an induced path on t vertices. In summer 2014 only the case $k = 4, t = 6$ remained open for all $k \geq 4$ and all $t \geq 6$. Huang conjectures that 4-colorability of P_6-free graphs can be decided in polynomial time. This conjecture has shown to be true for the class of $(P_6, banner)$-free graphs by Huang [11] and for the class of (P_6, C_5)-free graphs by Chudnovsky et al. [6]. In this paper we show that the conjecture also holds for the class of $(P_6, bull, Z_2)$-free graphs, for the class of $(P_6, bull, kite)$-free graphs, and for the class of $(P_6, chair)$-free graphs.

1 Introduction

We use [1] for terminology and notation not defined here and consider finite and simple graphs only.

Let G be a graph. An *induced subgraph* of G is a graph H such that $V(H) \subseteq V(G)$, and $uv \in E(H)$ if and only if $uv \in E(G)$ for all $u, v \in V(H)$. Given graphs G and F we say that G *contains* F if F is isomorphic to an induced subgraph of G. We say that a graph G is F-free, if it does not contain F.

A graph G is called *k-colorable*, if its vertices can be colored with k colors so that adjacent vertices obtain distinct colors. Since the k-colorability problem is NP-complete for every $k \geq 3$, it is natural to ask for the complexity of this problem when a certain induced subgraph H is forbidden.

We assume that $P \neq NP$. When we say that an algorithm runs "in polynomial time" or a problem can be solved "in polynomial time", we always mean "polynomial time as a function of the number of vertices of the input graph".

Theorem 1.1 *[12] For every $k, g \geq 3$, the k-colorability problem for graphs with no cycles of length at most g is NP-complete*

Applying this with $g = |V(H)|$ we obtain the following:

Theorem 1.2 *[12] Let H be a graph containing a cycle. For every $k \geq 3$, the k-colorability problem for H-free graphs remains NP-complete.*

Furthermore, constructing the line graph $L(G)$ for a given graph G, the following Theorem has been shown (cf. [3]).

Theorem 1.3 *[12] For every $k \geq 3$, the k-colorability problem for claw-free graphs is NP-complete.*

Consequently, we obtain

Theorem 1.4 *[12] Let H be a graph containing a claw. For every $k \geq 3$, the k-colorability problem for H-free graphs remains NP-complete.*

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Theorem 1.5 *[12] Let $k \geq 3$ be an integer, and H be a graph. If the k-colorability problem for H-free graphs can be solved in polynomial time, then every component of H is a path.*

2 Forbidden induced paths

In this section we briefly summarize what is currently known about the complexity of k-colorability with a forbidden induced path. Since P_4-free graphs are perfect, k-colorability can be solved in polynomial time for P_k-free graphs for $2 \leq k \leq 4$.

It has been shown in ([16],[15]) that 3-colorability can be solved in polynomial time for P_5-free graphs and for P_6-free graphs, respectively. Then it was proved in [9] that k-colorability can be solved in polynomial time for P_5-free graphs for all $k \geq 4$. Recently, major progress has been obtained ([11],[10]).

Theorem 2.1 *[11]*
1. The k-colorability problem is NP-complete for the class of P_t-free graphs for all $k \geq 5$ and $t \geq 6$.
2. The 4-colorability problem is NP-complete for the class of P_t-free graphs for all $t \geq 7$.

Theorem 2.2 *([4],[5])*
The 3-colorability problem can be solved in polynomial time for P_7-free graphs.

Hence, in summer 2014, the following two cases remained open:

(i) The complexity of 3-colorability for P_t-free graphs where $t \geq 8$, and

(ii) the complexity of 4-colorability for P_6-free graphs.

Two surveys on coloring graphs with forbidden induced subgraphs have been published [8,15].

3 Known results for P_6-free graphs

Randerath, Schiermeyer and Tewes [16] have studied the 4-colorability problem for (P_6, K_3)-free graphs.

Theorem 3.1 *[16] Every (P_6, K_3)-free graph is 4-colorable and there is a polynomial time algorithm for 4-coloring such graphs.*

Huang [11] has mentioned that K_3 can be replaced by the supergraph Z_1 (cf. Figure 1), which is also known as the paw.

Theorem 3.2 *[11] The 4-colorability problem can be solved in polynomial time for the class of (P_6, Z_1)-free graphs.*

Lozin and Rautenbach [13] considered graphs without long induced paths and obtained the following result.

Theorem 3.3 *[13] The 4-colorability problem can be solved in polynomial time for the class of $(P_6, K_{1,r})$-free graphs for any $r \geq 3$.*

Recently there has been some further progress.

Theorem 3.4 *[11] The 4-colorability problem can be solved in polynomial time for the class of $(P_6, banner)$-free graphs.*

Theorem 3.5 *[6] The 4-colorability problem can be solved in polynomial time for the class of (P_6, C_5)-free graphs.*

4 Main results

The starting point for our research has been Theorem 3.2 and the question, whether Z_1 can be replaced by a supergraph.

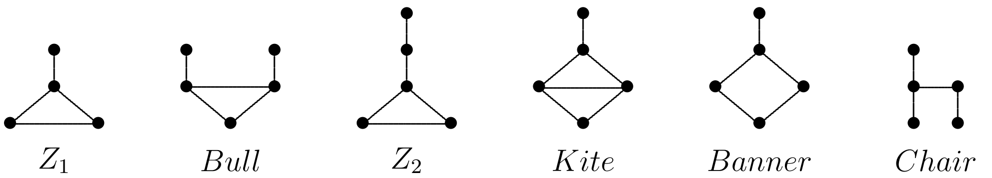

Fig. 1. The graphs Z_1, *Bull*, Z_2, *Kite*, *Banner*, and *Cricket*.

For the three supergraphs *bull*, Z_2 and *kite* we have obtained the following results.

Theorem 4.1 *The 4-colorability problem can be solved in polynomial time for the class of $(P_6, bull, Z_2)$-free graphs.*

Theorem 4.2 *The 4-colorability problem can be solved in polynomial time for the class of $(P_6, bull, kite)$-free graphs.*

Our next main result concerns Theorem 3.3 and the question, whether $K_{1,3}$ can be replaced by a supergraph. Here we have been able to show that in fact $K_{1,3}$ can be replaced by the chair (cf. Figure 1).

Theorem 4.3 *The 4-colorability problem can be solved in polynomial time for the class of $(P_6, chair)$-free graphs.*

5 Our proof approach

In this section we give a sketch of our proof approach.

We may assume that G contains a K_3 and an induced C_5, denoted by C, with vertex set $V(C) = \{v_1, v_2, v_3, v_4, v_5\}$ and edge set $E(C) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1\}$. Let $N^i(C)$ denote the set of all vertices of G which have distance i from C for $i \geq 1$.

The vertices of $N^1(C)$ can have the following neighbourhood structures on C :

- **Type 1:** w is adjacent to exactly one vertex v_i - set $M_{1,i}$
- **Type 2a:** w is adjacent to exactly two consecutive vertices v_i, v_{i+1} - set $M_{2a,i}$
- **Type 2b:** w is adjacent to exactly two non consecutive vertices v_i, v_{i+2} - set $M_{2b,i}$
- **Type 3a:** w is adjacent to exactly three consecutive vertices v_i, v_{i+1}, v_{i+2} - set $M_{3a,i}$
- **Type 3b:** w is adjacent to exactly three vertices v_i, v_{i+1}, v_{i+3} - set $M_{3b,i}$
- **Type 4:** w is adjacent to exactly four consecutive vertices $v_i, v_{i+1}, v_{i+2}, v_{i+3}$ - set $M_{4,i}$
- **Type 5:** w is adjacent to all five vertices v_1, v_2, v_3, v_4, v_5 - set M_5

Next we consider a 4-precoloring of C. Using the 2-Satisfiability approach described in [7] and applied in ([16],[15]), we can test in polynomial time, whether this 4-precoloring can be extended to a 4-coloring containg all vertices of Type 2a, Type 3a, Type 3b, Type 4, and Type 5. To make this approach work also for vertices of Type 1 and Type 2b, we search for a constant number of vertices, which we include in our set S of 4-precolored vertices. Clearly, also vertices in $N^2(C), N^3(C)$ and $N^4(C)$ have to be treated. Here we sometimes make use of cutsets, where clique cutsets and independent cutsets are very useful. We manage to show, that this branching concept only leads to a polynomial number of subproblems. Taking into account all these steps, the polynomial time complexity follows. □

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