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# The Maximum Independent Set Problem in Subclasses of $S_{i,j,k}$-Free Graphs

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**Abstract**

We consider the Maximum Independent Set (MIS) problem, which is known to be NP-hard in general, in subclasses of $S_{i,j,k}$-free graphs, where $S_{i,j,k}$ is the graph consisting of three induced paths of lengths $i$, $j$, $k$ with a common initial vertex. We revise the complexity of the problem. Then by two general approaches, augmenting graph and $\alpha$-redundant vertex, some graph classes, for which we have polynomial solutions, are obtained.

*Keywords:* Maximum Independent Set, Augmenting Graph, $\alpha$-Redundant Vertex

# 1 Introduction

In this paper, we consider the problem in hereditary graph classes, more precisely, under forbidden induced subgraph conditions. Graph classes, for which the problem is known to be NP-hard, are called MIS-*hard*. Classes, for which we have a polynomial solution for the problem, are called MIS-*easy*. Alekseev [1] observed the following result.

**Theorem 1.1** *[1] Given a finite graph set $\mathcal{F}$, let $\mathcal{X}$ be a hereditary graph class such that every graph of $\mathcal{X}$ is $\mathcal{F}$-free. If $\mathcal{F} \cap \mathcal{S} = \emptyset$, where $\mathcal{S}$ is the graph class in which every connected component of every graph is of the form $S_{i,j,k}$, then $\mathcal{X}$ is MIS-hard.*

For a single forbidden subgraph, Alekseev [2] showed that $S_{1,1,2}$-free graph class is MIS-easy. So far, in the case $i, j, k \geq 1$ and $i + j + k \geq 5$, there exists the MIS-easiness only for subclasses of $S_{i,j,k}$-free graphs. Some examples are ($S_{1,2,5}$,banner)-free graphs [10] and subclasses of $S_{1,2,2}$-free graphs and $S_{2,2,2}$-free graphs [9]. It motivates us to investigate on subclasses of $S_{i,j,k}$-free graphs.

# 2 Augmenting Graph

Augmenting graph is a general technique to solve the MIS problem and can be described as follows. Given a graph $G$ and an independent set $S$, a bipartite graph $H = (W, B, E)$ is called augmenting for $S$ if (i) $W \subset S$, $B \subset V(G)\backslash S$, (ii) $N(B) \cap (S\backslash W) = \emptyset$, and (iii) $|B| > |W|$.

**Theorem 2.1** *[8] An independent set $S$ in a graph $G$ is maximum if and only if there is no augmenting graph for $S$.*

This theorem suggests the following general approach to find a maximum independent set in a graph $G$. Starting with any independent set $S$ (may be empty) in $G$, as long as $S$ admits an augmenting graph $H$, apply $H$-augmentation to $S$. Clearly, the problem of finding augmenting graphs is generally NP-hard, as the MIS problem is NP-hard. For a polynomial time solution for some graph class, one has to solve the two following problems:

**(P1)** Find a complete list of augmenting graphs in the class under consideration.

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**(P2)** Develop polynomial time algorithms for detecting all augmenting graphs in the class.

Let $\text{apple}_p^k$ denote the graph consisting of a chordless cycle of length $p$ and an induced path of length $k$ having one end-vertex in common with the cycle. In the case $p = 4$, we call it $\text{banner}_k$ and in the case $k = 1$, we write simply $\text{apple}_p$. A graph $G$ is an $(k, m)$-extended-chain if $G$ is a tree and contains two vertices $a, b$ such that there exists an induced path $P \subset G$ connecting $a, b$, every vertex of $G - P$ is of distance at most $k - 1$ from either $a$ or $b$, and every vertex of $G - P$ has no neighbor in $P$ except possibly $a$ or $b$ and every vertex of $G$ is of degree at most $m - 1$. We extend Theorem 4.8 in [8] to the following result.

**Lemma 2.2** *For any three integers $k, l,$ and $m$ such that $4 \leq 2k \leq l$ and $m \geq 3$, in ($S_{2,2k,l}$,$\text{apple}_4^l$, $\text{apple}_6^l$,...,$\text{apple}_{2k+2}^l$,$K_{1,m}$)-free graphs, there are only finitely many minimal augmenting graphs which are neither augmenting $(2k, m)$-extended-chains nor $\text{apple}_{2p}$. Moreover, if $H$ is of the form augmenting $(2k, m)$-extended-chain, then every white vertex is of degree two.*

Note that we can always find augmenting graphs belonging to some finite set in polynomial time. Concerning the case an augmenting graph $H$ contains a vertex of degree large enough, Dabrowski et al. [5] proved that $H$ must contain an induced $K_{m,m}$ or a $\text{tree}_m$, where $\text{tree}_m$ is the graph consisting of $m$ $P_3$'s sharing an end-vertex. For the first case, using the concept of redundant sets of Lozin and Milanič, we obtain the following result.

**Lemma 2.3** *In ($\text{banner}_2$, $K_{3,m} - e$, $M_m$)-graphs, the problem of finding a minimal augmenting graph containing an induced $K_{m,m}$ is polynomially reducible to the problem of finding an augmenting graph of the form $K_{m,m+1}$.*

Remind the result of Hertz and Lozin [8] that augmenting graphs of the form $K_{m,m+1}$ can be found in $\text{banner}_2$-free graphs polynomially. Then we extend Theorem 2 in [7] to obtain the following result.

**Lemma 2.4** *Given integers $l$ and $m$, where $l$ is even, an ($S_{2,l,l}$,$\text{banner}_l$,$R_l^1$, $R_l^2$, $R_l^3$, $R_l^4$, $R_l^5$)-free graph $G$, and an independent set $S$ in $G$, one can determine whether $S$ admits an augmenting $(l, m)$-extended-chain or an augmenting apple in polynomial time.*

Together with previous observations, it leads us to the following results.

**Theorem 2.5** *Given three integers $k, l,$ and $m$ such that $4 \leq 2k \leq l$, the fowllowing graph classes are MIS-easy:*

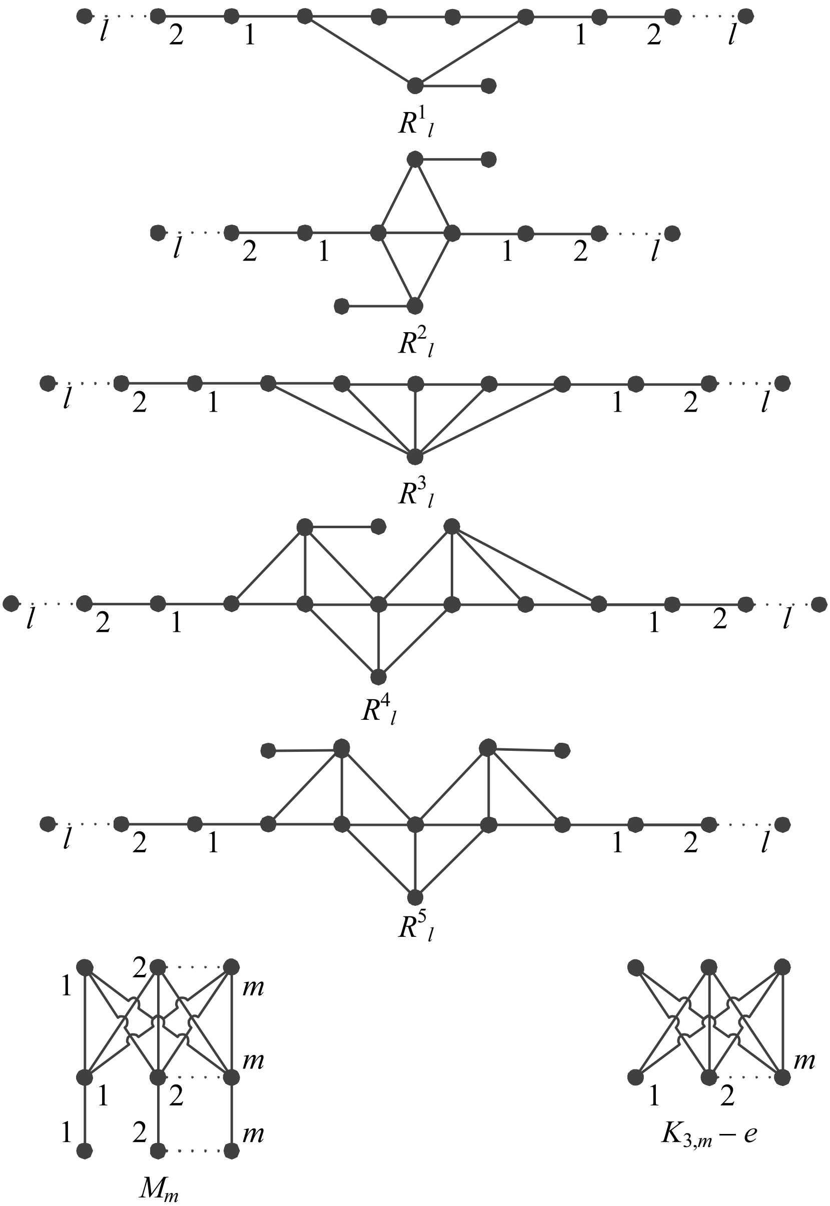


Fig. 1. $R_l^1$,..., $R_l^5$, $M_m$, and $K_{3,m} - e$

(i) *($S_{2,k,l}$, $banner_l$, $apple_6^l$, $apple_8^l$, ..., $apple_{2k+2}^l$, $R_l^1$, $R_l^2$, $R_l^3$, $R_l^4$, $R_l^5$, $K_{m,m}$, $tree_m$)-free graphs and*

(ii) *($S_{2,k,l}$, $banner_2$, $apple_6^l$, $apple_8^l$, ..., $apple_{2k+2}^l$, $R_l^1$, $R_l^2$, $R_l^3$, $R_l^4$, $R_l^5$, $K_{3,m-e}$, $M_m$, $tree_m$)-free graphs.*

**Corollary 2.6** *Given three integers $k, l, m$, the following graph classes are MIS-easy:*

(i) *($S_{1,k,l}$,banner$_l$,apple$^l_6$,apple$^l_8$, ...,apple$^l_{2k+2}$, $R^2_l$, $K_{m,m}$,tree$_m$)-free graphs,*

(ii) *($S_{1,k,l}$,banner$_2$,apple$^l_6$,apple$^l_8$, ...,apple$^l_{2k+2}$, $R^2_l$, $K_{3,m-e}$, $M_m$,tree$_m$)-free graphs,*

(iii) *($S_{2,2,l}$,banner$_l$,$R^3_l$, $R^4_l$, $R^5_l$, $K_{m,m}$,tree$_m$)-free graphs, and*

(iv) *($S_{2,2,l}$,banner$_2$,$R^3_l$, $R^4_l$, $R^5_l$, $K_{3,m-e}$, $M_m$,tree$_m$)-free graphs.*

These results are generalizations of similar results for ($P_l$, $K_{m,m}$,tree$_m$)-free graphs [5], ($S_{1,2,l}$,banner,$K_{1,m}$)-free graphs and ($S_{1,2,3}$,banner$_k$,$K_{1,m}$)-free graphs [8], ($P_5$, $K_{2,m} - e$)-free graphs [3], and ($P_5$, $K_{3,3} - e$)-free graphs [6]. In [5], the authors also described the methods of finding minimal augmenting graphs containing a tree$_m$ in $S_{1,1,3}$-free graphs, which leads us to the following result.

**Theorem 2.7** *The ($S_{1,1,3}$,banner$_2$,$K_{3,m} - e$,tree$_m$)-free graph class is MIS-easy.*

## 3 $\alpha$-Redundant Vertex

A vertex is called $\alpha$-redundant if its removal does not change the independence number [4]. Consider an induced $S_{i,j,k}$ in an $S_{i+1,j,k}$-free graph, where $1 \leq i \leq j, k$; $a$ is the vertex of degree three; and $(a = b_0, b_1, \ldots, b_i)$ is the path of length $i$, we argue that $b_{i-1}$ is $\alpha$-redundant and can be removed from the graph. Using this argument consecutively on results mentioned in the previous section, we obtain the following theorem.

**Theorem 3.1** *Given two integers $k, m$, the following graph classes are MIS-easy:*

(i) *($S_{2,2,5}$,banner,apple$_5$,...,apple$_8$)-free graphs,*

(ii) *($S_{3,3,5}$,banner,apple$_5$,...,apple$_9$)-free graphs,*

(iii) *($S_{2,2,k}$,banner,apple$_5$,...,apple$_{k+3}$,tree$_m$)-free graphs,*

(iv) *($S_{3,3,k}$,banner,apple$_5$,...,apple$_{k+4}$,tree$_m$)-free graphs,*

(v) *($S_{2,k,k}$, $R^2_k$,banner,apple$_5$,...,apple$_{k+3}$,tree$_m$)-free graphs,*

(vi) *($S_{3,k,k}$, $R^2_k$,banner,apple$_5$,...,apple$_{k+4}$,tree$_m$)-free graphs, and*

(vii) *($S_{k,l,l}$,apple$_3$,...,apple$_{k+l+1}$)-free graph.*

## 4 Conclusion

Nearly two decades after the MIS-easiness has been showed for claw-free ($S_{1,1,1}$-free) graphs independently in 1980 by Minty [12] and Sbihi [13], Alekseev [2] has shown a similar result for fork-free ($S_{1,1,2}$-free) graphs. So far, there exists no further result for $S_{i,j,k}$-free graphs in the case $i, j, k \geq 1$ and $i + j + k \geq 5$ but only in subclasses [9,10].
It is also worth to notice that Minty, Sbihi, Alekseev, and Lozin and Milanič also used augmenting graph for their proofs. By this technique, we have obtained polynomial solutions for some subclasses of $S_{1,1,3}$-free, $S_{2,k,l}$-free graphs. Then these results have been extended to $S_{2,2,k}$-free, $S_{3,3,k}$-free, $S_{2,k,k}$-free, $S_{3,3,k}$-free, $S_{3,k,k}$, $S_{k,l,l}$-free graphs by $\alpha$-redundant vertex technique. To the best of our knowledge, there are still not many results in these areas. These results also extend previous work of several authors.

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