Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in DISCRETE MATHEMATICS

www.elsevier.com/locate/endm

Nowhere-zero 5-flows

Eckhard Steffen [1]

Paderborn Institute for Advanced Studies in Computer Science and Engineering
Paderborn University
Paderborn, Germany

Giuseppe Mazzuoccolo [2]

Dipartimento di Scienze Fisiche, Informatiche e Matematiche
Università di Modena e Reggio Emilia
Modena, Italy

Abstract

We prove that every cyclically 6-edge-connected cubic graph with oddness at most 4 has a nowhere-zero 5-flow. Therefore, a possible minimum counterexample to the 5-flow conjecture has oddness at least 6.

Keywords: nowhere-zero flows, 5-flow conjecture, cubic graphs

1 Introduction

An integer nowhere-zero k-flow on a graph G is an assignment of a direction and a value of $\{1, \dots, (k-1)\}$ to each edge of G such that the Kirchhoff's law is satisfied at every vertex of G. Seymour [4] proved that every bridgeless graph

[1] Email: `es@upb.de`

[2] Email: `giuseppe.mazzuoccolo@unimore.it`

has a nowhere-zero 6-flow. So far this is the best approximation to Tutte's famous 5-flow conjecture, which is equivalent to its restriction to cubic graphs.

Conjecture 1.1 **([7])** *Every bridgeless graph has a nowhere-zero 5-flow.*

A snark is a cubic bridgeless graph which is not 3-edge-colorable. A classical parameter to measure how far a cubic bridgeless graph is from being 3-edge-colorable is its *oddness*. The oddness, denoted by $\omega(G)$, of a bridgeless cubic graph G is the minimum number of odd circuits in a 2-factor of G. The following three statements are equivalent: (i) $\omega(G) = 0$; (ii) G is 3-edge-colorable; (iii) G has a nowhere-zero 4-flow. Hence, a possible counterexample to the 5-flow-conjecture is a snark. It is easy to see that snarks with oddness 2 have a nowhere-zero 5-flow. If the cyclic connectivity of a cubic graph G is big in terms of its oddness, then G has a nowhere-zero 5-flow.

Theorem 1.2 **([5])** *Let G be a bridgeless cubic graph with cyclic connectivity k. If $k \geq \frac{5}{2}\omega(G) - 3$, then G has a nowhere-zero 5-flow.*

By a result of Kochol [2], it suffices to prove the 5-flow conjecture for cyclically 6-edge-connected snarks. There are infinitely many cyclically 6-edge-connected snarks, but no snark with cyclic connectivity greater than 6 is known. The following is our main theorem.

Theorem 1.3 *Let G be a cyclically 6-edge-connected cubic graph. If $\omega(G) \leq 4$, then G has a nowhere-zero 5-flow.*

We deduce:

Corollary 1.4 *If G is a possible minimum counterexample to the 5-flow conjecture, then*

- *G is a cubic graph [4].*
- *G is cyclically 6-edge connected [2].*
- *the cyclic connectivity of G is at most $\frac{5}{2}\omega(G) - 4$ [5].*
- *G has girth at least 11 [3].*
- *G has oddness at least 6.*
- *$G - e$ has circular flow number 5 for each $e \in E(G)$ [6].*

2 Sketch of the proof of Theorem 1.3

We combine structural and coloring properties of cubic graphs to prove that there is no minimum counterexample to the statement of Theorem 1.3. Suppose to the contrary that there is one, say G. We start with a specific 4-edge-coloring of G. By adding edges between the odd circuits of a 2-factor of G we construct graphs M_1, M_2, M_3 which have a nowhere-zero 4-flow. From this we deduce specific partitions of $V(G)$ into two sets A and B with $|A| = |B|$, which give us information of the distribution of the colors on critical edge-cuts of G. It follows, that at least one these partitions of the vertices of G corresponds to a nowhere-zero 5-flow on G by Theorem 2.1. We give some details in the following:

2.1 *Balanced valuation*

A *balanced valuation* of a graph G is a function f from the vertex set $V(G)$ into the real numbers, such that $|\sum_{v\in X} f(v)| \leq |\partial_G(X)|$ for all $X \subseteq V(G)$. Jaeger proved the following fundamental theorem.

Theorem 2.1 **([1])** *Let G be a graph with orientation D and $k \geq 3$. Then G has a nowhere-zero k-flow if and only if there is a balanced valuation f of G with $f(v) = \frac{k}{k-2}(2d^+_{D(G)}(v) - d_G(v))$, for all $v \in V(G)$.*

Hence, a cubic graph G has a nowhere-zero 5-flow if and only if there is a balanced valuation of G with values in $\{\pm\frac{5}{3}\}$.

2.2 *Canonical coloring and flows*

Let G be a bridgeless cubic graph, and $\mathcal{F}_2$ be a 2-factor of G with odd circuits $C_1, \ldots, C_{2t}$, and even circuits $C_{2t+1}, \ldots, C_{2t+l}$ ($t \geq 0$, $l \geq 0$), and let $\mathcal{F}_1$ be the complementary 1-factor.

A *canonical* 4-edge-coloring, denoted by c, of G with respect to $\mathcal{F}_2$ colors the edges of $\mathcal{F}_1$ with color 1, the edges of the even circuits of $\mathcal{F}_2$ with 2 and 3, alternately, and the edges of the odd circuits of $\mathcal{F}_2$ with colors 2 and 3 alternately, but one edge which is colored 0. Then, there are precisely $2t$ vertices $z_1, \ldots, z_{2t}$ where color 2 is missing (that is, no edge which is incident to z_i has color 2).

The subgraph which is induced by the edges of colors 1 and 2 is union of even circuits and t paths P_i of odd length and with $z_1, \ldots, z_{2t}$ as ends. Without loss of generality we can assume that P_i has ends z_{2i-1} and z_{2i}, for $i \in \{1, \ldots, t\}$.

Let M_G be the graph obtained from G by adding two edges f_i and f'_i between z_{2i-1} and z_{2i} for $i \in \{1, \ldots, t\}$. Extend the previous edge-coloring to a proper edge-coloring of M_G by coloring f'_i with color 2 and f_i with color 4. Let $C'_1, \ldots, C'_s$ be the cycles of the 2-factor of M_G induced by the edges of colors 1 and 2 ($s \geq t$). In particular, C'_i is the even circuit obtained by adding the edge f'_i to the path P_i, for $i \in \{1, \dots, t\}$. Finally, for $i \in \{1, \dots, t\}$ let C''_i be the 2-circuit induced by the edges f_i and f'_i. We construct a nowhere-zero 4-flow on M_G as follows:

- for $i \in \{1, \dots, 2t + l\}$ let (D_i, φ_i) be a nowhere-zero flow on the directed circuit C_i with $\varphi_i(e) = 2$ for all $e \in E(C_i)$;
- for $i \in \{1 \dots, s\}$ let (D'_i, φ'_i) be a nowhere-zero flow on the directed circuit C'_i with $\varphi'_i(e) = 1$ for all $e \in E(C'_i)$;
- for $i \in \{1, \dots, t\}$ let (D''_i, φ''_i) be a nowhere-zero flow on the directed circuit C''_i (choose D''_i such that f'_i receives the same direction as in D'_i) with $\varphi''_i(e) = 1$ for all $e \in \{f_i, f'_i\}$.

Then,

$$(D, \varphi) = \sum_{i=1}^{2t+l} (D_i, \varphi_i) + \sum_{i=1}^{s} (D'_i, \varphi'_i) + \sum_{i=1}^{t} (D''_i, \varphi''_i)$$

is the desired nowhere-zero 4-flow on M_G.

2.3 Flow partition

By Theorem 2.1, there is a balanced valuation $w(v) = 2(2d^+_{D(M_G)}(v) - d_{M_G}(v))$ of M_G. It holds that $|2d^+_{D(M_G)}(v) - d_{M_G}(v)| = 1$, and hence, $w(v) \in \{\pm 2\}$ for all vertices v. The vertices of M_G, and therefore, of G as well, are partitioned into two classes $A = \{v | w(v) = -2\}$ and $B = \{v | w(v) = 2\}$. We call the elements of A (B) the white (black) vertices of G, respectively.

Definition 2.2 Let G be a bridgeless cubic graph and $\mathcal{F}_2$ a 2-factor of G. A partition of $V(G)$ into two classes A and B constructed as above with a canonical 4-edge-coloring c, the 4-flow (D, φ) on M_G and the induced balanced valuation w of M_G is called a **flow partition** of G w.r.t. $\mathcal{F}_2$. The partition is denoted by $P_G(A, B)(= P_G(A, B, \mathcal{F}_2, c, (D, \varphi), w))$.

Lemma 2.3 ([5]) *Let G be a bridgeless cubic graph and $P_G(A, B)$ be a flow partition of $V(G)$ which is induced by a canonical nowhere-zero 4-flow with respect to an edge-coloring c. Let x, y be the two vertices of an edge e. If $e \in c^{-1}(1) \cup c^{-1}(2)$, then x and y belong to different classes, i.e. $x \in A$ if and only if $y \in B$.*

From a flow partition $P_G(A, B)(= P_G(A, B, \mathcal{F}_2, c, (D, \varphi), w))$ we easily obtain a flow partition $P_G(A', B')(= P_G(A', B', \mathcal{F}_2, c, (D', \varphi'), w'))$ such that the colors on the vertices of P_i are switched. Let (D', φ') be the nowhere-zero 4-flow on M_G obtained by using the same 2-factor $\mathcal{F}_2$, the same 4-edge-coloring c of G and the same orientations for all circuits, but for one $i \in \{i, \dots, t\}$ use opposite orientation of C_i' and C_i'' with respect to the one selected in (D, φ).

Lemma 2.4 *Let G be a bridgeless cubic graph and $P_G(A, B)$ be the flow partition which is induced by the nowhere-zero 4-flow (D, φ). If $P_G(A', B')$ is the flow partition induced by the nowhere-zero 4-flow (D', φ'), then $A \setminus V(P_i) = A' \setminus V(P_i)$, $B \setminus V(P_i) = B' \setminus V(P_i)$, $A \cap V(P_i) = B' \cap V(P_i)$ and $B \cap V(P_i) = A' \cap V(P_i)$.*

2.4 Critical edge-cuts

For $i \in \{1, 2, 3, 4\}$ let C_i be an odd circuit of a minimum 2-factor of G, and let z_i be the vertex of C_i, where color 2 is missing (w.r.t. the canonical 4-edge-coloring of G). Let $S \subset V(G)$ and $a = |S \cap A|$, $b = |S \cap B|$. An edge-cut $\partial(S)$ is *critical* if it separates two subsets of cardinality 2 of $\{z_1, \dots, z_4\}$, and $\frac{5}{3}|a - b| > |\partial_G(S)|$. We show that at least one of the induced flow partitions does not contain a critical edge-cut.

References

[1] Jaeger, F., *Balanced valuations and flows in multigraphs*, Proc. Amer. Math. Soc. **55** (1975), 237-242

[2] Kochol, M. *Reduction of the 5-flow conjecture to cyclically 6-edge-connected snarks*, J. Combin. Theory, Ser. B **90** (2004), 139-145

[3] Kochol, M. *Smallest counterexample to the 5-flow conjecture has girth at least eleven*, J. Combin. Theory Ser. B **100** (2010), 381-389

[4] Seymour, P. D. *Nowhere-zero 6-flows*, J. Combin. Theory, Ser. B **30** (1981), 130-135

[5] Steffen, E. *Tutte's 5-flow conjecture for highly cyclically connected cubic graphs*, Discrete Math. **310** (2010), 385-389

[6] Steffen, E. *Nearly nowhere-zero r-flow graphs*, Discrete Math. **312** (2012), 2757-2759

[7] Tutte, W. T. *A contribution to the theory of chromatic polynomials*, Canadian J. Math. **6** (1954), 80-91