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On the random greedy F-free hypergraph process

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Abstract

Let F be a strictly k-balanced k-uniform hypergraph with $e(F) \geq |F| - k + 1$ and maximum co-degree at least two. The random greedy F-free process constructs a maximal F-free hypergraph as follows. Consider a random ordering of the hyperedges of the complete k-uniform hypergraph K_n^k on n vertices. Start with the empty hypergraph on n vertices. Successively consider the hyperedges e of K_n^k in the given ordering and add e to the existing hypergraph provided that e does not create a copy of F. We show that asymptotically almost surely this process terminates at a hypergraph with $\tilde{O}(n^{k-(|F|-k)/(e(F)-1)})$ hyperedges. This is best possible up to logarithmic factors.

Keywords: random greedy, hypergraph, F-free process.

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1 Introduction

1.1 Results

Fix a k-uniform hypergraph F. We study the following random greedy process, which constructs a maximal F-free k-uniform hypergraph. Assign a birthtime which is uniformly distributed in $[0,1]$ to each hyperedge of the complete k-uniform hypergraph K_n^k on n vertices. Start with the empty hypergraph on n vertices at time $p = 0$. Increase p and each time that a new hyperedge is born, add it to the hypergraph provided that it does not create a copy of F (edges with equal birthtime are added in any order). Denote the resulting hypergraph at time p by $R_{n,p}$.

The random greedy graph process (i.e. the case when $k = 2$) has been studied for many graphs. The initial motivation (see for example [8]) was to study the Ramsey number $R(3,t)$. Indeed, the best current lower bounds on $R(3,t)$ were obtained via the study of the triangle-free process ([5], [10]). Osthus and Taraz [11] gave an upper bound on the number of edges in the graph $R_{n,1}$ when F is strictly 2-balanced, showing that a.a.s. $R_{n,1}$ has maximum degree $O(n^{1-(|F|-2)/(e(F)-1)}(\log n)^{1/(\Delta(F)-1)})$. (Here a.a.s. stands for 'asymptotically almost surely', i.e. for the property that an event occurs with probability tending to one as n tends to infinity.) Results for the cases when $F = C_4$ and $F = K_4$ were obtained independently by Bollobás and Riordan [7]. Bohman and Keevash [4] showed that a.a.s. $R_{n,1}$ has minimum degree $\Omega(n^{1-(|F|-2)/(e(F)-1)}(\log n)^{1/(e(F)-1)})$ whenever F is strictly 2-balanced and conjectured that this gives the correct order of magnitude. Improved upper bounds have been obtained for some graphs. For instance, the number of edges has been determined asymptotically when F is a cycle ([3], [5], [10], [12], [14]) and when $F = K_4$ ([15], [16]). Picollelli [13] determined asymptotically the number of edges when F is a diamond, i.e. the graph obtained by removing one edge from K_4. Note that this graph is not strictly 2-balanced.

Much less is known about the process when F is a k-uniform hypergraph and $k \geq 3$. The only known upper bound is due to Bohman, Mubayi and Picollelli [6], who studied the F-free process when F is a k-uniform generalisation of a graph triangle (with an application to certain Ramsey numbers). We obtain a generalisation of the upper bound in [11] to strictly k-balanced hypergraphs. Here we say that a k-uniform hypergraph F is *strictly k-balanced* if $|F| \geq k+1$ and for all proper subgraphs $F' \subsetneq F$ with $|F'| \geq k+1$ we have

$$\frac{e(F)-1}{|F|-k} > \frac{e(F')-1}{|F'|-k}.$$

We also need the following definition. Given a hypergraph H and $i \in \mathbb{N}$, we define the *maximum i-degree* of H by

$$\Delta_i(H) := \max\{d_H(U) : U \subseteq V(H), |U| = i\},$$

where $d_H(U)$ is the number of hyperedges in H containing U.

Theorem 1.1 *Let $k \in \mathbb{N}$ be such that $k \geq 2$. Let F be a strictly k-balanced k-uniform hypergraph which has v vertices and $h \geq v - k + 1$ hyperedges. Suppose $\Delta_{k-1}(F) \geq 2$. Then there exists a constant c such that a.a.s.*

$$\Delta_{k-1}(R_{n,1}) < t \qquad \textit{where} \qquad t := cn^{1-\frac{v-k}{h-1}}(\log n)^{\frac{3}{\Delta_{k-1}(F)-1}-\frac{1}{h-1}}. \tag{1}$$

In particular, a.a.s. $R_{n,1}$ has at most tn^{k-1} hyperedges.

Note that Theorem 1.1 applies, for example, to all k-uniform cliques K_v^k on $v \geq k+1$ vertices and more generally to all balanced complete ℓ-partite k-uniform hypergraphs with $\ell \geq k$ and more than k vertices. Theorem 1.1 also applies when F is a k-uniform tight cycle. Loose cycles, however, do not satisfy the co-degree condition in Theorem 1.1. We conjecture that the upper bound on the number of hyperedges holds in this case also.

Bennett and Bohman [2] studied a random greedy independent set algorithm in certain quasi-random hypergraphs. This algorithm finds a maximal independent set by choosing vertices uniformly at random and adding them to the existing set provided they do not create a hyperedge. Note that we can define an $e(F)$-regular hypergraph H whose vertex set is $E(K_n^k)$ and whose hyperedges correspond to all copies of F in K_n^k. In this case, the random greedy independent set process on H is exactly the F-free process. Their result can be applied in the context of the F-free process to show that if F is a strictly k-balanced k-uniform hypergraph and every vertex of F lies in at least two hyperedges, then a.a.s. $R_{n,1}$ has $\Omega(n^{k-(|F|-k)/(e(F)-1)}(\log n)^{1/(e(F)-1)})$ hyperedges. Up to logarithmic factors, this matches the upper bound given in Theorem 1.1.

1.2 An open question

There are many natural open questions related to the random greedy F-free process. One of which would be to generalise Theorem 1.1 by finding an upper bound on the number of steps in the random greedy independent set process studied in [2].

The random greedy independent set process can also be applied to study arithmetic progression free sets. Suppose $k, n \in \mathbb{N}$. The kAP-free process generates a subset I of $\mathbb{Z}_n$ which does not contain an arithmetic progression of length k as follows. The elements of $\mathbb{Z}_n$ are ordered uniformly at random. Each is then, in turn, added to the set I if it does not create a k term arithmetic progression. So this is another instance of the random greedy independent set algorithm, this time on the hypergraph with vertex set $\mathbb{Z}_n$ whose hyperedges are all arithmetic progressions of length k. When n is prime, Bennett and Bohman [2] showed that a.a.s. the kAP-free process generates a kAP-free set I of size $\Omega(n^{(k-2)/(k-1)}(\log n)^{1/(k-1)})$. It would be interesting to obtain a corresponding upper bound on I. (Note that an upper bound on the number of steps in the random greedy independent set process would imply an upper bound for the kAP-free process.)

1.3 Sketch of the argument

Rather than studying the random greedy process itself, we are able to prove Theorem 1.1 by obtaining precise information about the random binomial hypergraph $H_{n,p}$. (This idea was first used in [11].) More precisely, write $H_{n,p}$ for the random binomial k-uniform hypergraph on n vertices with hyperedge probability p, i.e., each hyperedge is included in $H_{n,p}$ with probability p, independently of all other hyperedges. We write $H^-_{n,p}$ for the hypergraph formed by removing all copies of F from $H_{n,p}$. Note that $H_{n,p}$ can also be viewed as the random hypergraph consisting of all hyperedges with birthtime at most p. Thus, for all $p \in [0,1]$ we have $H^-_{n,p} \subseteq R_{n,p} \subseteq R_{n,1}$. We will always assume that K^k_n, $H_{n,p}$, $H^-_{n,p}$ and $R_{n,p}$ use the vertex set $[n]$.

The proof of Theorem 1.1 proceeds as follows. We first identify the largest point p where we can still use $H_{n,p}$ to approximate the behaviour of $H^-_{n,p}$ (i.e. for this p, only a small proportion of edges of $H_{n,p}$ lie in a copy of F). Now let U be a set of $k-1$ vertices in F such that $d_F(U) = \Delta_{k-1}(F)$. Let $\hat{F}$ be the subgraph of F obtained by deleting all those hyperedges which contain U. Let t be as in (1). Suppose for a contradiction that there exists a $(k-1)$-set V of degree t in $R_{n,1}$ and let T be the neighbourhood of V in $R_{n,1}$. We will show that in this case we would almost certainly find a copy α of $\hat{F}$ in $H^-_{n,p}[T \cup V]$ which maps U to V. Since $H^-_{n,p} \subseteq R_{n,1}$, α would also be a copy of $\hat{F}$ in $R_{n,1}[T \cup V]$ which maps U to V. But this actually yields a copy of F in $R_{n,1}$, a contradiction. So a.a.s. $\Delta_{k-1}(R_{n,1}) < t$. It is perhaps surprising that for our analysis the order of hyperedges added after this critical point p is irrelevant.

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