Available online at www.sciencedirect.com

ScienceDirect

ELSEVIER

Electronic Notes in DISCRETE MATHEMATICS

www.elsevier.com/locate/endm

Upper bounds on the numbers of 1-factors and 1-factorizations of hypergraphs

Anna Taranenko [1,2]

Sobolev Institute of Mathematics and Novosibirsk State University
Novosibirsk, Russia

Abstract

A hypergraph $G = (X, W)$ is called d-uniform if each hyperedge w is a set of d vertices. A 1-factor of a hypergraph G is a set of hyperedges such that every vertex of the hypergraph is incident to exactly one hyperedge from the set. A 1-factorization of G is a partition of all hyperedges of the hypergraph into disjoint 1-factors.

The adjacency matrix of a d-uniform hypergraph G is the d-dimensional (0,1)-matrix of order $|X|$ describing sets of vertices of G such that they make a hyperedge.

We estimate the number of 1-factors of uniform hypergraphs and the number of 1-factorizations of complete uniform hypergraphs by means of permanents of their adjacency matrices.

Keywords: hypergraph, 1-factor, 1-factorization, permanent, multidimensional matrix

[1] The work is supported by the Russian Science Foundation (grant 14–11–00555)

[2] Email: `taattg@mail.ru`

1 Introduction

Let $G = (V, E)$ be a graph on n vertices. The adjacency matrix $M(G)$ of G is the (0,1)-matrix of order n such that the entry $m_{i,j}$ equals one if and only if the vertices i and j are adjacent. A 1-factor (perfect matching) of the graph G is a 1-regular subgraph that has the same vertex set as G. A 1-factorization of G is a partition of edges of the graph into disjoint 1-factors.

It is wellknown that the number of 1-factors of a balanced bipartite graph is equal to the permanent of its biadjacency matrix (rows of this matrix correspond to the first partite set, and the columns, to the second). At the same time, this number is equal to the square root of the permanent of the adjacency matrix. In [1], Alon and Friedland proved that the number of 1-factors of any graph is not greater than the square root of the permanent of its adjacency matrix.

Also, permanents can be used for the estimation of the number of the complete graph 1-factorizations [7]. Let $\Phi(n)$ denote the number of 1-factorizations of the complete graph K_n on n vertices. Then

$$\left((1+o(1))\frac{n}{4\mathrm{e}^2}\right)^{\frac{n^2}{2}} \leq \Phi(n) \leq \left((1+o(1))\frac{n}{\mathrm{e}^2}\right)^{\frac{n^2}{2}}.$$

The lower bound was obtained by Cameron in [4]. This proof requires the van der Waerden conjecture which was proved by Egorychev in [5] and Falikman in [6]. The upper bound follows from Bregman's theorem for the permanent of (0,1)-matrices [3] and from the result of [1].

There exist several bounds on the number of 1-factorizations of other graphs. For example, for d-regular bipartite graphs we have the following result proved by Schrijver in [8]:

Theorem 1.1 *Let G be a d-regular bipartite graph on $2n$ vertices. Then the number of 1-factorizations of G is not less than* $\left(\frac{d!^2}{d^d}\right)^n$.

Here we estimate the number of 1-factors and 1-factorizations of uniform hypergraphs by means of permanents of multidimensional matrices.

Let $n, d \in \mathbb{N}$, and let $I_n^d = \{(\alpha_1, \ldots, \alpha_d) : \alpha_i \in \{1, \ldots, n\}\}$. Then a *$d$-dimensional matrix A of order n* is an array $(a_\alpha)_{\alpha \in I_n^d}$, $a_\alpha \in \mathbb{R}$.

For a d-dimensional matrix A of order n, denote by $D(A)$ the set of its diagonals

$$D(A) = \left\{(\alpha^1, \ldots, \alpha^n) | \alpha^i \in I_n^d, \forall i \neq j \ \rho(\alpha^i, \alpha^j) = d\right\},$$

where ρ is the Hamming distance (the number of positions at which the corresponding entries are different). Then the *permanent* of a matrix A is

$$\mathrm{per} A = \sum_{p \in D} \prod_{\alpha \in p} a_\alpha.$$

Also, recall some definitions on hypergraphs.

The pair $G = (X, W)$ is called a *d-uniform hypergraph* on n vertices with vertex set X and hyperedge set W if $|X| = n$ and each hyperedge $w \in W$ is a set of d vertices. A hypergraph G is called *simple* if it has no multiple hyperedges. The *degree* of a vertex $x \in X$ in a hypergraph G is the number of hyperedges containing x.

A *1-factor* of a hypergraph G is a set of hyperedges such that every vertex of the hypergraph is incident to exactly one hyperedge from the set. A *1-factorization* of G is a partition of all hyperedges of the hypergraph into disjoint 1-factors. A d-uniform hypergraph $G = (X, W)$ in which all vertices have the degree d is called a *d-factor*.

The *incidence matrix* of a hypergraph G is the $|X| \times |W|$ matrix $(b_{i,j})$ such that $b_{i,j} = 1$ if the vertex x_i and the hyperedge w_j are incident, and 0 otherwise. The *adjacency matrix* $M(G)$ of a d-uniform hypergraph G is the d-dimensional (0,1)-matrix of order n such that an entry m_α equals one if and only if the vertices with numbers from α make a hyperedge of G.

2 An upper bound on the number of 1-factors of hypergraphs

Let G be a simple d-uniform hypergraph on n vertices. Denote by $\varphi(G)$ the number of 1-factors of G. It is clear that if there exists a 1-factor of the hypergraph G, then the number of vertices n is divisible by d. Therefore, below we will consider only n multiple of d.

It is quite easy to prove that the number of 1-factors of a hypergraph G is not greater than the permanent of its adjacency matrix. We obtain the following theorem, that strengthens this bound:

Theorem 2.1 *Let G be a simple d-uniform hypergraph on n vertices, and let d divide n. Define the function $\mu(n, d)$ such that $\mu(n, 2) = \mu(n, 3) = 1$ for all integer n and*

$$\mu(n, d) = \frac{d!^{2n}}{d^{dn} d!^{n/d}}$$

for all $d \geq 4$. Then the number of 1-factors of the hypergraph G satisfies

$$\varphi(G) \leq \left(\frac{\mathrm{per} M(G)}{\mu(n,d)}\right)^{1/d}.$$

Corollary 2.2 *The number of 1-factors of a simple d-uniform hypergraph is not greater than the dth root of the permanent of its adjacency matrix:*

$$\varphi(G) \leq (\mathrm{per} M(G))^{1/d}.$$

Corollary 2.3 *Let $G = (X, W)$ be a simple d-uniform hypergraph on n vertices, and let the vertex $x_i \in X$ have the degree r_i. Then the number of 1-factors of the hypergraph G satisfies*

$$\varphi(G) \leq \left(\frac{(d-1)!^n}{\mu(n,d)} \prod_{i=1}^{n} r_i\right)^{1/d}.$$

The proof of Theorem 2.1 is based on ideas of paper [1]. Let us give a sketch of proof of the theorem.

Denote by $\mathfrak{F}(G)$ the set of all ordered d-tuples of 1-factors of G, where d-tuples can contain identical 1-factors. It is clear that $|\mathfrak{F}(G)| = \varphi^d(G)$.

Let $f \in \mathfrak{F}(G)$ be an ordered d-tuple of 1-factors. Consider the d-uniform hypergraph F on n vertices such that its hyperedge set is exactly the set of all hyperedges of the d-tuple f, and the multiplicities of hyperedges from F and f are the same. By construction, F is a 1-factorable d-factor. Denote by $\Phi(F)$ the number of all 1-factorizations of F (i.e., the number of all d-tuples $f \in \mathfrak{F}(G)$ corresponding to F).

Let w be a hyperedge of a hypergraph G. An arbitrary ordering of vertices of a hyperedge w is said to be an *orientation* of w.

An *orientation* of a hypergraph G is the set of orientations of all its hyperedges, where the number of orientations of a hyperedge w equals to its multiplicity. A *proper orientation* of a hypergraph G is an orientation such that there are no vertices having the same position in different orientations of hyperedges. Let $\delta(G)$ be the set of all proper orientations of G, and let $\Delta(G) = |\delta(G)|$ be the cardinality of this set.

Let F_1 and F_2 be 1-factorable d-factors. Note that if the hyperedge sets of F_1 and F_2 are the same (taking into account the multiplicity of hyperedges), then $\delta(F_1) = \delta(F_2)$. If the hyperedge sets of F_1 and F_2 are different, then all orientations of F_1 and F_2 will be different too, and $\delta(F_1) \cap \delta(F_2) = \emptyset$.

Therefore, all d-tuples from $\mathfrak{F}(G)$ can be divided into classes such that d-tuples from one class induce the same d-factor F, the cardinality of each class equals $\Phi(F)$, and the sets of proper orientations for different classes are disjoint.

Now we give the key statement for the proof of Theorem 2.1:

Proposition 2.4 *Let F be a 1-factorable d-factor. Then*

$$\Phi(F) \leq \frac{\Delta(F)}{\mu(n,d)}.$$

By this proposition, it is quite easy to prove Theorem 2.1.

Let G be a simple d-uniform hypergraph. Put $\gamma(G) = \bigcup \delta(F)$, where the union is over all d-factors F costructed by all $f \in \mathfrak{F}(G)$. Note that the set of entries of the adjacency matrix $M(G)$, whose indices make a proper orientation from $\gamma(G)$, forms a unity diagonal in $M(G)$. Consequently, $|\gamma(G)| \leq \mathrm{per} M(G)$. The following is a simple corollary to Proposition 2.4:

Corollary 2.5 *Let G be a simple d-uniform hypergraph on n vertices. Then*

$$|\mathfrak{F}(G)| \leq \frac{|\gamma(G)|}{\mu(n,d)}.$$

Proof of Theorem 2.1 Recall that $\varphi^d(G) = |\mathfrak{F}(G)|$. By Corollary 2.5, $|\mathfrak{F}(G)| \leq \frac{|\gamma(G)|}{\mu(n,d)}$. Also, we know that $|\gamma(G)|$ is not greater than the permanent of the adjacency matrix $M(G)$. Therefore,

$$\varphi(G) \leq \left(\frac{\mathrm{per} M(G)}{\mu(n,d)}\right)^{1/d}.$$

Let us describe the main ideas of the proof of Proposition 2.4 now. We state firstly that it is sufficient to consider only connected hypergraphs F.

Lemma 2.6 *Suppose that for all connected 1-factorable d-factors F on n vertices we have $\Phi(F) \leq \frac{\Delta(F)}{\mu(n,d)}$. Then this inequality holds for disconnected hypergraphs too.*

In the proof of Proposition 2.4 we use the concept of bipartite representation of a hypergraph. For a hypergraph $G = (X, W)$, the *bipartite representation* of G is the bipartite graph $B(G) = (X, W; E)$ with the vertex set $X \cup W$, and E is the edge set; the vertex $x \in X$ is adjacent to the vertex $w \in W$ in $B(G)$ if and only if the vertex x is incident to the edge w in G. Note that the biadjacency matrix of $B(G)$ coincides with the incidence matrix of G.

Any bipartite graph can be considered as a bipartite representation of some hypergraph. If G is a d-factor, then its bipartite representation $B(G)$ is a d-regular graph, and each row and each column of the adjacency matrix of $B(G)$ contains d ones.

At the next step, we associate the numbers of 1-factorizations and proper orientations of a d-factor F with the numbers of proper decompositions and proper edge colorings of its bipartite representation.

Let $G = (V, E)$ be a graph. A *proper edge coloring* with k colors of the graph G is an assignment of 'colors' to the edges of the graph so that no two adjacent edges have the same color. If G is a d-regular bipartite graph on $2n$ vertices, then each proper edge coloring of G with d colors is equivalent to some 1-factorization of G.

Let $B = (X, Y; E)$ be a d-regular bipartite graph with the partite sets X and Y such that $|X| = |Y| = n$, and let d divide n. A *proper decomposition* of the partite set Y is a decomposition of Y into disjoint equinumerous subsets $Y_1, \ldots, Y_d$ such that the neighborhood of each Y_i (the union of neighborhoods of y over all $y \in Y_i$) is equal to X. In other words, each vertex $x \in X$ is adjacent to exactly one vertex from each Y_i.

Remind that the d-factor F in Proposition 2.4 may contain multiple hyperedges that correspond in $B(F)$ to the vertices $w \in W$ with identical neighborhoods. Suppose that there are k different hyperedges in the hypergraph F, and let the ith hyperedge have the multiplicity l_i, $i = 1, \ldots, k$. Put $R(F) = \prod\limits_{i=1}^{k} l_i!$.

The following lemmas describe the connection between these structures in a d-factor F and in its bipartite representation $B(F)$:

Lemma 2.7 *Let F be a 1-factorable d-factor, and let $B(F)$ be its bipartite representation. Denote by $P(B)$ the number of proper edge colorings of $B(F)$ with d colors. Then*

$$\Delta(F) = P(B)/R(F).$$

Lemma 2.8 *Let F be a 1-factorable d-factor, and let $B(F)$ be its bipartite representation. Denote by $T(B)$ the number of proper decompositions of the partite set W of $B(F)$. Then*

$$\Phi(F) = T(B)/R(F).$$

Therefore, to obtain Proposition 2.4 it is sufficient to prove the next lemma, which follows from Theorem 1.1:

Lemma 2.9 *Let $B = (X, Y; E)$ be a d-regular connected bipartite graph on $2n$ vertices, and let d divide n. Then $T(B) \leq \frac{P(B)}{\mu(n,d)}$.*

3 An upper bound on the number of 1-factorizations of complete hypergraphs

Denote by G_n^d the complete d-uniform hypergraph on n vertices, that is, the hyperedge set of G_n^d is the set of all d-element subsets of the vertex set. Let $M(G_n^d)$ be the adjacency matrix of this graph. Recall that if a hypergraph has a 1-factor, then n is a multiple of d. By Baranyai's theorem [2], this condition is sufficient for the existence of a 1-factorization of complete hypergraphs. Denote by $\Phi(n,d)$ the number of 1-factorizations in the hypergraph G_n^d.

It is not hard to prove the following upper bound on the number of 1-factorizations:

Proposition 3.1 *The number of 1-factorizations of the hypergraph G_n^d satisfies*

$$\Phi(n,d) \leq \left((1+o(1))\frac{n^{d-1}}{(d-1)!}\right)^{\frac{n^d}{d!}} \quad \text{as } n \to \infty.$$

We prove the following theorem, that strengthens this bound:

Theorem 3.2 *The number of 1-factorizations of the complete d-uniform hypergraph G_n^d on n vertices satisfies*

$$\Phi(n,d) \leq \left((1+o(1))\frac{n^{d-1}}{\mu(n,d)^{1/n}\mathrm{e}^d}\right)^{\frac{n^d}{d!}} \quad \text{as } n \to \infty.$$

Corollary 3.3 *If $d \leq 3$, then the number of 1-factorizations of the complete d-uniform hypergraphs G_n^2 and G_n^3 on n vertices satisfies*

$$\Phi(n,d) \leq \left((1+o(1))\frac{n^{d-1}}{\mathrm{e}^d}\right)^{\frac{n^d}{d!}} \quad \text{as } n \to \infty.$$

If $d \geq 4$, then the number of 1-factorizations of G_n^d satisfies

$$\Phi(n,d) \leq \left((1+o(1))\left(\frac{d}{\mathrm{e}}\right)^d \frac{n^{d-1}}{d!^{2-1/d}}\right)^{\frac{n^d}{d!}} \quad \text{as } n \to \infty.$$

Our reasoning for complete hypergraphs is similar to the proof for graphs, but instead of the result of [1] and Bregman's theorem we use Theorem 2.1 and the result of [9].

References

[1] Alon, N., and S. Friedland, *The maximum number of perfect matchings in graphs with a given degree sequence*, The Electronic J. of Combinatorics **15** (2008).

[2] Baranyai, Zs., *On the factorization of the complete uniform hypergraph*, Infinite and Finite Sets, Proc. Coll. Keszthely, Colloquia Math. Soc. Janos Bolyai **10** (1973), 91–107.

[3] Bregman, L. M., *Some properties of nonnegative matrices and their permanents*, Soviet Math. Dokl. **14** (1973), 945–949.

[4] Cameron, P. J., "Parallelisms of complete designs," Lecture Note Ser. **23**, London Math. Soc., Cambridge Univ. Press, Cambridge, 1976.

[5] Egorychev, G. P., *Proof of the van der Waerden conjecture for permanents*, Siberian Math. J. **22** (1981), 854–859.

[6] Falikman, D. I., *A proof of the van der Waerden conjecture regarding the permanent of a doubly stochastic matrix*, Math. Notes Acad. Sci. USSR **29** (1981), 475–479.

[7] Linial, N., and Z. Luria, *An upper bound on the number of Steiner triple systems*, Random Structures and Algorithms **43**, Issue **4** (2013), 399–406.

[8] Schrijver, A., *Counting 1-factors in regular bipartite graphs*, J. of Comb. Theory **B72**, Issue 1 (1998), 122–135.

[9] Taranenko, A. A., *Upper bounds on the permanent of multidimensional (0,1)-matrices*, Siberian Electronic Math. Reports **11** (2014), 958–965.