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Towards a flow theory for the dichromatic number

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Abstract

We transfer the ideas of analyzing the chromatic number of a graph using nowhere-zero-coflows and -flows to digraphs and the dichromatic number.

Keywords: dichromatic number, colorings, flows

1 Introduction

In [4] Victor Neumann-Lara introduced the dichromatic number $\vec{\chi}(D)$ of a digraph $D = (V, A)$ as the smallest integer k such that the vertices of D can be colored with k colors such that each color class induces a directed acyclic graph.

We give a characterization of the dichromatic number in terms of coflows of the digraph and develop a flow theory dual to this.

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2 Notation

Let $D = (V, A)$ be a directed graph. A mapping $f = (f_1, f_2) : A \to \mathbb{Z}^2$ is called a NEUMANN-LARA-*flow* or NL-flow for short, if both components of f satisfy Kirchhoff's law of flow conversation

$$\forall v \in V : \sum_{a \in \delta^-(v)} f_i(a) = \sum_{a \in \delta^+(v)} f_i(a)$$

and furthermore

$$f_1(a) = 0 \Rightarrow f_2(a) > 0.$$

An NL-flow is an NL-k flow, if

$$\forall a \in A : |f_1(a)| < k.$$

A mapping $f^* = (f_1^*, f_2^*) : A \to \mathbb{Z}^2$ is called an NL-coflow for short, if for each cycle C of the underlying undirected graph

$$\sum_{a \in C^+} f_i^*(a) = \sum_{a \in C^-} f_i^*(a) \tag{1}$$

where C^+ and C^- denote the arcs of C that are traversed in forward resp. backward direction and furthermore

$$f_1^*(a) = 0 \Rightarrow f_2^*(a) > 0.$$

An NL-coflow is an NL-k-coflow, if

$$\forall a \in A : |f_1^*(a)| < k.$$

Theorem 2.1 *Let $D = (V, A)$ be a loopless connected directed graph. Then D has an NL-k-coflow if and only if it has dichromatic number at most k.*

Proof. Let $f^* : A \to \mathbb{Z}^2$ be an NL-k-coflow. We define a coloring of c as follows. Choose an arbitrary vertex $v \in V$ which receives color zero $c(v) = 0$. Now let w be another vertex and P_1 be a (not necessarily directed) v-w-path in D. Then we define the color of w as

$$\tilde{c}(w) = \sum_{a \in P_1^+} f_1^*(a) - \sum_{a \in P_1^-} f_1^*(a)$$

where P_1^+ and P_1^- denote the arcs of P_1 that are traversed in positive resp. negative direction and claim that this value is independent of the chosen path. Namely, if P_2 is another such path, then the concatenation of P_1 and P_2 traversed backwards is a closed tour and hence can be decomposed into circuits of D. Since f_1^* is a coflow in D, f_1^* sums to zero on all of these circuits. Hence

$$\sum_{a \in P_1^+} f_1^*(a) - \sum_{a \in P_1^-} f_1^*(a) - \sum_{a \in P_2^+} f_1^*(a) + \sum_{a \in P_2^-} f_1^*(a) = 0$$

and $\tilde{c}$ is well defined. Now in order to get c in the proper range we set $c(h) = \tilde{c}(h) \bmod k$.

We are left to verify that the color classes of this coloring induce acyclic subdigraphs. Assume we had a directed cycle in one color class. Then $f_1^* \equiv 0$ on this cycle C and thus $f_2^* > 0$ on C, hence

$$\sum_{a \in C_+} f_2^*(a) > 0 = \sum_{a \in \emptyset} f_2^*(a) = \sum_{a \in C_-} f_2^*(a)$$

contradicting the definition of a coflow.

On the other hand if we have a coloring c with colors $\{0, \ldots, k-1\}$ such that each color class induces an acyclic directed graph, we define an NL-k-coflow as follows. If $a = (v, w) \in A$ is an arc of D we put $f_1^*(a) = c(w) - c(v)$. Since f_1^* this way is defined by a potential it vanishes on every cycle and hence satisfies (1). Let A_1 denote the set of arcs which receive an non-zero f_1^*

$$A_1 := \{a \in A \mid f_1^*(a) \neq 0\}.$$

Since each color class induces an acyclic directed graph, already $D \setminus A_1$ must be acyclic. Hence using topological sort we find an ordering $V = \{v_1, \ldots, v_n\}$ of its vertices such that forall $a = (v_i, v_j) \in A \setminus A_1$ we have $i < j$. Hence putting

$$f_2^* = \sum_{i=1,\ldots,n-1} \vec{1}_{\partial(\{v_1,\ldots,v_i\})},$$

where $\vec{1}_{\partial(\{v_1,\ldots,v_i\})}$ denotes the directed characteristic function of the cut defined by $\{v_1, \ldots, v_i\}$, we find a function that vanishes on all cycles and is strictly positive on $A \setminus A_1$. Note, that for $a = (v_i, v_j) \in A$ we get $f_2^*(a) = j - i$. □

3 Planar Digraphs

In [3] Neumann-Lara conjectured that the dichromatic number of an orientation of a planar simple graph is bounded by 2. Clearly, an NL-k-flow in a bridgeless planar digraph D is an NL-k-coflow in its dual D^* and vice versa. Hence, two-colorability of every orientation of a planar graph is equivalent to the existence of an NZ-2-flow in every planar digraph whose underlying graph is three edge connected. The support of f_1 of an NL-2-flow must be an even subgraph E, i.e. an edge disjoint union of not necessarily directed cycles. Contracting E f_2 becomes a strictly positive integer vector in the cycle space of D/E, which can be decomposed into a sum of not necessarily disjoint directed cycles. On the other hand, if contracting an even subgraph we have a strictly positive flow, this yields a flow in the original graph, which is strictly positive outside of the even subgraph. On the even subgraph we find a flow using only ± 1. Hence Neumann-Lara's conjecture is equivalent to

Conjecture 3.1 *Let $D = (G, A)$ be a three edge connected planar digraph. There exists an even subgraph $E \subseteq A$ such that D/E is strongly connected.*

Lemma 3.2 *Every orientation of the Petersen graph admits an NL-2-flow.*

Proof. It suffices to show that there always exist two vertex disjoint 5-cycles, the complement of a perfect matching, such that the matching edges are not all oriented the same way with respect to the cycles. Starting with the pentagon and the pentagram we are done, if the complementary matching edges are oriented not all the same way. Therefore, and by symmetry, we may assume that all edges are directed from the pentagram to the pentagon. Now considering the red circuits in Figure 1 and the complementary edge to the uppermost vertex we are done, if not all matching edges are oriented towards the upper cycle, indicated in blue. Using the symmetry of the Petersen graph and rotating the configuration we find two cycles the contraction of which leaves a strongly connected graph. □

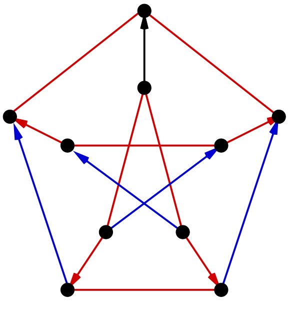

Fig. 1. Any orientation of the Petersen graph has an NL-2-flow

By Tutte's 4-flow conjecture [5] the Petersen graph is the only cographical obstruction to 4-colorability. Since the Petersen graph is not an obstruction to the existence of an NL-2-flow we are tempted to conjecture

Conjecture 3.3 *Let $D = (G, A)$ be a three edge connected digraph. There exists an even subgraph $E \subseteq A$ such that D/E is strongly connected.*

4 Oriented Matroids

There is a natural way to generalize the above to oriented matroids the same way as Tutte's coloring and flow theory for regular matroids was generalized to oriented matroids by Hochstättler, Nešetřil and, later, Hochstättler and Nickel ([1],[2]).

Assume we are given an oriented matroid $\mathcal{O}$ on a finite set E represented by its covectors. By $\mathcal{D}$ we denote its set of cocircuits and for $D \in \mathcal{D}$ by $\vec{\chi}(D)$ its signed characteristic function. Recall that the chromatic number $\chi(\mathcal{O})$ of an oriented matroid is defined as the smallest k such that the lattice of coflows

$$\mathcal{F}^*(\mathcal{O}) := \left\{ \sum_{D \in \mathcal{D}} \lambda_D \vec{\chi}_D \mid \lambda_D \in \mathbb{Z} \right\}. \tag{2}$$

contains a coflow $f^* \in \mathcal{F}^*(\mathcal{O})$ such that

$$\forall e \in E : 0 < |f(e)| < k.$$

An NL-coflow in an oriented matroid is a tuple $(f^*, f^+) \in \mathcal{F}^*(\mathcal{O}) \times \mathcal{O}$ such that

$$\forall e \in E : f^*(e) = 0 \Rightarrow f^+_e = +.$$

The dichromatic number $dichr(\mathcal{O})$ then is defined as the smallest k, such that there exists an NL-coflow (f^*, f^+) such that

$$\forall e \in E : |f^*(e)| < k.$$

Note that replacing each element in an oriented matroid by a pair of antiparallel elements the dichromatic number of the constructed oriented matroid is the chromatic number of the original one. Hence, as in the graphic case the dichromatic number is a proper generalization of the chromatic number. Clearly the chromatic number of the underlying reorientation class of an oriented matroid is always an upper bound for the dichromatic number. Furthermore, the dichromatic number is 1 if and only if the oriented matroid is acyclic, meaning that $\mathcal{O}$ contains the all +-vector.

We observe that the dichromatic number of an oriented matroid which has a cospanning cocircuit is bounded by 2.

Proposition 4.1 *Let $\mathcal{O}$ be a uniform oriented matroid of rank $r \geq 1$ on n elements, which has an independent hyperplane, i.e. a flat F in the underlying matroid such that $rg(F) = r - 1 = |F|$. Then $dichr(\mathcal{O}) \leq 2$.*

Proof. $D := E \setminus F$ is a cocircuit of size $n - r + 1$. In $\mathcal{O}$ we choose one of its two orientations. Since F is independent, D is cospanning. Hence, for each $e_i \in E \setminus \operatorname{supp}(D)$ for $0 \leq i \leq r-1$ we can choose a cocircuit D^i such that $\operatorname{supp}(D^i) \setminus \operatorname{supp}(D) = \{e_i\}$ consists of a single new element and $D^i(e_i) = +$. Now we set

$$X = D \circ D^1 \circ \cdots \circ D^{r-1}.$$

Then X is a covector without zeroes which is positive outside of $\operatorname{supp}(D)$. Hence, setting $f^* = \vec{\chi}(D)$ yields the NL-2-coflow (f^*, X). $\square$

Corollary 4.2 *If $\mathcal{O}$ is a uniform oriented matroid, the orientation of a paving matroid or the cographic matroid of a Hamiltonian graph, then $dichr(\mathcal{O}) \leq 2$.*

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