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Regularity lemmas in a Banach space setting

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Abstract

Szemerédi's regularity lemma is a fundamental tool in extremal graph theory, theoretical computer science and combinatorial number theory. Lovász and Szegedy [7] gave a Hilbert space interpretation of the lemma and an interpretation in terms of compactness of the space of graph limits. In this paper we prove several compactness results in a Banach space setting, generalising results of Lovász and Szegedy [7] as well as a result of Borgs, Chayes, Cohn and Zhao [2].

Keywords: Regularity lemma, graph limits, L^p graphon, Banach space.

1 Introduction

1.1 The regularity lemma

Szemerédi's regularity lemma is a fundamental tool in extremal graph theory, theoretical computer science and combinatorial number theory. The lemma has many interpretations, variations and extensions.

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Very roughly the lemma says something of the form: for each $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that the vertex set of any graph can be partitioned into at most k parts, such that for 'almost' all pairs of parts the edges between that pair of parts behaves 'almost' like a random bipartite graph, where 'almost' depends on ε. The weak regularity lemma of Frieze and Kannan [4] weakens the requirements of the partition in the regularity lemma and measures the error of approximation with respect to the cut norm. From the perspective of the adjacency matrix of a graph this means that one approximates this matrix with a bounded sum of cut matrices (in particular, this gives a low rank approximation) such that their difference is small with respect to the cut norm. This is exactly the point of view we take in this paper: we want to find various types of low rank approximations to matrices and tensors, when measured in a particular norm.

Our work is inspired by the work of Lovász and Szegedy [7] and Borgs, Chayes, Cohn and Zhao [2] relating the compactness of the space of graph limits to Szemerédi's regularity lemma. We refer to the book by Lovász [5] for more details on graph limits. In [6] Lovász and Szegedy used the weak version of the regularity lemma [4] to assign a limit object to a convergent sequence of dense graphs. This limit object is no longer a graph, but a symmetric measurable function $W : [0,1]^2 \to [0,1]$, called a *graphon*. In [7] Lovász and Szegedy showed that the space of graphons, equipped with the cut metric is compact, interpreting this result as an analytical form of the regularity lemma. Their compactness result implies various kinds of regularity lemmas varying from weak to very strong. It has recently been extended by Borgs, Chayes, Cohn and Zhao [2] to the space of $\mathbb{R}$-valued functions W with bounded p-norm, the *L^p-graphon space* (for any fixed $p > 1$).

1.2 Compactness

We will now describe the compactness of the graphon space, which is denoted by $\mathcal{W}$, more precisely, after which we state the main result of the present paper.

Let $W : [0,1]^2 \to \mathbb{R}$. Consider for $p, q \in [1, \infty]$, W as a kernel operator $W : L^p([0,1]) \to L^q([0,1])$. The $p \mapsto q$-operator norm is defined by

$$\|W\|_{p\mapsto q} = \sup_{\|f\|_p=1} \|Wf\|_q = \sup_{\|f\|_p=1, \|g\|_{q^*}=1} \int_{[0,1]^2} W(x,y)g(x)f(y)d\lambda,$$

where $\|\cdot\|_s$ denotes the s-norm on the space $L^s([0,1])$ and where $q^* = q/(q-1)$. The norm $\|\cdot\|_{\infty\mapsto 1}$ is equivalent to the cut norm. Call $W \sim W'$ if for each

$\varepsilon > 0$ there exists a measure preserving bijection $\tau : [0,1] \to [0,1]$ such that $\|W - \tau W'\|_{\infty\mapsto 1} \leq \varepsilon$. Then the result of Lovász and Szegedy [7] can be stated as follows:

$$\text{the space } (\mathcal{W}, \|\cdot\|_{\infty\mapsto 1})/\sim \text{ is compact.} \tag{1}$$

The result of Borgs, Chayes, Cohn and Zhao [2] then says that we can replace $\mathcal{W}$ with the symmetric functions in $L^p([0,1]^2)$ of norm at most 1 for any fixed $p > 1$, which is denoted by $B(L^p([0,1]^2)$. We show that in (1) we can also replace the norm $\|\cdot\|_{\infty\mapsto 1}$ by the norm $\|\cdot\|_{q\mapsto \frac{q}{q-1}}$:

Theorem 1.1 *Let $p \in (1,\infty]$ and let q be such that $q > \frac{p}{p-1}$. Then the space $(B(L^p([0,1]^2)), \|\cdot\|_{q\mapsto\frac{q}{q-1}})/\sim$ is compact.*

Here $W \sim W'$ if for each $\varepsilon > 0$ there exists a measure preserving bijection $\tau : [0,1] \to [0,1]$ such that $\|W - \tau W'\|_{q\mapsto\frac{q}{q-1}} \leq \varepsilon$. Note that the theorem fails to be true when $p = \frac{q}{q-1}$.

We derive Theorem 1.1 from a general result about compact orbit spaces in Banach spaces, replacing in (1) the space $\mathcal{W}$ by a weakly compact subset of a Banach space X, the relation $\sim$ by an equivalence relation obtained from a subgroup of the group of automorphisms of X and the norm $\|\cdot\|_{\infty\mapsto 1}$ by an operator-type norm, cf. Theorem 2.1.

Our method is based on work of the author and Schrijver [11], in which the compactness result of Lovász and Szegedy was extended to a general Hilbert space setting, putting emphasis on the possibility of using different norms than the cut norm and the use of groups and moreover using a different method of proof. Consequently, our proof of Theorem 2.1 does not use the martingale convergence theorem. Thus it yields a different proof of the compactness result of Borgs, Chayes, Cohn and Zhao [2].

This extended abstract is based on [10]. We refer the reader for proofs, which have been mostly omitted, extensions to higher order tensors and further details to [10].

2 Compact orbit spaces in Banach spaces

Before we can state our result, we need some definitions. Let $X = (X, \|\cdot\|)$ be a normed space. By $B(X)$ we denote unit ball in X. Let $R \subseteq B(X^*)$, where X^* is the dual space of X. We define a seminorm $\|\cdot\|_R$ and pseudo metric d_R on X by

$$\|x\|_R := \sup_{r\in R} |r(x)| \qquad d_R(x,y) := \|x-y\|_R$$

for $x, y \in X$.

For a pseudo metric space (X, d) let $\mathrm{Aut}(X)$ denote the group of invertible maps $g : X \to X$ that preserve d. Let G be a subgroup of $\mathrm{Aut}(X)$. Define a pseudo metric d/G on X by

$$(d/G)(x, y) := \inf_{g \in G} d(x, gy)$$

for $x, y \in X$. Note that since $(d/G)(x, y)$ is just equal to the distance between the G-orbits of x and y, this implies that d/G is indeed a pseudo metric. For our purposes it is sometimes more convenient to work with $(X, d/G)$ than with X/G, but note that $(X, d/G)$ is compact if and only if X/G is compact. Recall that a (pseudo) metric space is called *totally bounded* of for each $\varepsilon > 0$ it can be covered with finitely many balls of radius ε.

For a subset Y of a linear space X and $k \in \mathbb{N}$ we define

$$k \cdot Y := \{y_1 + \ldots + y_k \mid y_i \in Y\}.$$

Note that when Y is convex, $k \cdot Y$ is just equal to kY. Let X be a normed space and let $W \subset X$. Let H be a Hilbert space We call W *H-small*, if there exists a contractive linear map $T : H \to X$ and a function $c : (0, \infty) \to \mathbb{N}$ such that $W \subset c(\varepsilon)T(B(H)) + \varepsilon B(X)$ for each $\varepsilon > 0$. Note that T gives rise to a contractive linear map $T^* : X^* \to H^*$ defined by $f \mapsto (h \mapsto f(T(h)))$ for $f \in X^*$ and $h \in H$. When we talk about a H-small space we will implicitly assume the presence of the maps T, T^* and c.

Theorem 2.1 *Let $(X, \|\cdot\|)$ be a Banach space and let G be a subgroup of* $\mathrm{Aut}(X)$. *Let $R \subseteq B(X^*)$, be G-stable and let $W \subset X$ be a G-stable and weakly, or weakly sequentially compact set. Then*

(i) if $(W, d_R/G)$ is totally bounded, then $(W, d_R/G)$ is compact;

(ii) if W is H-small for some Hilbert space H, and if $(k \cdot (TT^*R), d_R/G)$ is totally bounded for each $k \in \mathbb{N}$, then $(W, d_R/G)$ is totally bounded and hence compact by (i).

Observe that when X is a Hilbert space, Theorem 2.1 reduces to [11, Theorem 2.1]. Theorem 2.1 can be proved using a similar method as has been used in [11] and we omit the proof.

3 Regularity lemmas

In [7], Lovász and Szegedy applied the compactness of the graphon space, cf. (1), to derive approximation results for graphons, cf. [7, Lemma 5.2]. This result implies several types of regularity lemmas varying from the weak regularity lemma [4], to the original lemma [12], to a 'super strong' variant [1]. See [5] for more details. We can derive something similar in our Banach space setting (we omit the proof):

Lemma 3.1 *Let $(X, \|\cdot\|)$ be a Banach space, let $G \subseteq \text{Aut}(X)$, let $R \subset B(X^*)$ and let $W \subset X$ be G-stable and suppose that $(W, d_R/G)$ is compact. Let for $k \in \mathbb{N}$, $Y_k \subset W$ be G-stable such that $Y := \cup_{k\in\mathbb{N}} Y_k$ is dense in W (w.r.t. $\|\cdot\|$). Let $h : (0, \infty) \times \mathbb{N} \to (0, 1)$ be any function. Then for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for any $w \in W$ there exists $w' \in W$ and $y \in Y_m$, with $m \leq n$, such that*

$$\|w - w'\|_R \leq h(\varepsilon, m) \quad \text{and} \quad \|w' - y\| \leq \varepsilon.$$

Since the norm $\|\cdot\|$ on X satisfies $\|X\| \leq \|x\|_R$ for each $x \in X$, taking $h(\varepsilon, m) = \varepsilon$ for all m, Lemma 3.1 implies several types of weak regularity lemmas by taking different choices of Y_k. When applying this to $L^p([0,1]^2)$, one can for example take Y_k to be the collection of functions that are constant on $V_i \times V_j$ for some partition $\{V_1, \ldots, V_k\}$ of $[0, 1]$

4 Proof sketch of Theorem 1.1

Let us first remark that the compactness of the L^p graphon space proved by Borgs, Chayes, Cohn and Zhao [2] follows easily from Theorem 2.1.

Indeed, let $p \in (1, \infty]$, let $W = B(L^p([0,1]^2)) \subset X = L^1([0,1]^2)$. (If $p \geq 2$ we take $X = L^2([0,1]^2)$.) Taking $R = \{\chi_{A\times B} \mid A, B \subset [0,1] \text{ measurable}\}$, makes $\|\cdot\|_R$ into the cut norm. As group G we take the group of measure preserving bijections $\phi : [0,1] \to [0,1]$. Then d_R/G is equal to $\delta_\square$, the cut metric. The compactness result from [2] can now be stated as follows:

$$(W, d_R/G) \text{ is compact.} \tag{2}$$

To see how (2) follows from Theorem 2.1, let $H = L^2([0,1]^2)$. Then W is H-small by [2] or see Lemma 4.1 below. Note that TT^* restricted to R is the identity. Since any measurable set A can be mapped onto any interval of length $\lambda(A)$ by a measure preserving bijection, cf. [8], it follows that $(k \cdot R, d_R/G)$ is compact (see [11] for details). It is an easy consequence of the Banach-Alaoglu theorem that W is weakly compact. Thus Theorem 2.1 (ii) now implies that $(W, d_R/G)$ is compact.

To prove Theorem 1.1 we need some additional results. First of all define for any $q \geq 1$, $R^q := \{f_1 \otimes f_2 \mid f_i \in B(L^q([0,1]))\}$ and note that for any $W \in L^q([0,1]^2$ we have, by Hölder's inequality, $\|W\|_{q \mapsto \frac{q}{q-1}} = \|W\|_{R^q}$.

Lemma 4.1 *Let (Ω, μ) be any probability space. Let $p > p' \geq 1$ and $\varepsilon > 0$. Then there exists a constant C such that $B(L^p(\Omega)) \subseteq CB(L^\infty(\Omega)) + \varepsilon B(L^{p'}(\Omega))$.*

Lemma 4.2 *Let $p > s \geq 1$ and let $\varepsilon > 0$. Then there exists a constant C such that $R^p \subseteq CR^\infty + \varepsilon R^s$.*

With these two lemmas (whose proof we omit), the proof proceeds as follows. We let $X = L^{q^*}([0,1]^2)$, and $W = B(L^p([0,1]^2))$. The group G is defined as above. As a consequence of (1) we know that $B(L^\infty([0,1]^2, d_{R^\infty}/G)$ is compact. (Note that this is also implied by (2).) We use this combined with Lemma 4.2 to show that $(B(L^\infty([0,1]^2)), d_{R^q}/G)$ is compact. Using Lemma 4.1 we then show that $(W, d_{R^q}/G)$ is totally bounded. The Banach-Alaoglu theorem is used to show that W is weakly compact. We can then invoke Theorem 2.1 (i) to conclude that $(W, d_{R^q}/G)$ is compact.

5 Algorithmic applications

So far we have obtained one algorithmic application of the results mentioned here. It is not so much a direct application of the regularity type lemmas, but more an application of the proof of Theorem 1.1. Combining Lemmas 4.1 and 4.2 with the sampling results of Borgs, Chayes, Lovász, Sós and Vesztergombi [3], we obtain randomised approximation algorithms for computing $\ell^q \mapsto \ell^{q/(q-1)}$- matrix norms. In particular, there exists a RPAS for dense matrices. See [9] for more details.

References

[1] Alon, N., E. Fischer, M. Krivelevich, and M. Szegedy: *Efficient testing of large graphs*, Combinatorica **20** (2000), 451–476.

[2] Borgs, C., J.T. Chayes, H. Cohn, and Y. Zhao, *An L^p theory of sparse graph convergence I: limits, sparse random graph models, and power law distributions*, arXiv preprint arXiv:1401.2906 (2014).

[3] Borgs, C., J.T. Chayes, L. Lovász, V.T. Sós, and K. Vesztergombi, *Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing*, Adv. Math. **219** (2008), 1801–1851.

[4] Frieze, A., and R. Kannan, *Quick approximation to matrices and applications*, Combinatorica **19** (1999), 175–220.

[5] Lovász, L., "Large Networks and Graph Limits", Amer. Math. Soc., Providence, R.I., 2012.

[6] Lovász, L., and B. Szegedy, *Limits of dense graph sequences*, J. Comb. Theory B **96** (2006), 933–957.

[7] Lovász, L., and B. Szegedy, *Szemerédi's Lemma for the analyst*, Geom. Funct. Anal. **17** (2007), 252–270.

[8] Nishiura,T., *Measure-Preserving Maps of* $\mathbb{R}^n$, Real Analysis Exchange **24** (1999), 837–842.

[9] Regts, G., *Approximation algorithms for matrix norms from regularity lemmas in Banach spaces*, in preparation.

[10] Regts, G., *Regularity lemmas in a Banach space setting*, arXiv preprint, arXiv.1502.04849, (2015).

[11] Regts, G., and A. Schrijver, *Compact orbit spaces in Hilbert spaces and limits of edge-colouring models*, to appear in Eur. J. Combin., arXiv preprint, arXiv.1210.2204 (2012).

[12] Szemerédi, E. *Regular partitions of graphs*, in: Problèmes combinatoires et théorie des graphes, (Proceedings Colloque International C.N.R.S., Paris-Orsay, 1976) [Colloques Internationaux du C.N.R.S. No **260**], Editions du C.N.R.S., Paris, 1978, pp. 399–401.