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Edge-decompositions of graphs with high minimum degree

Ben Barber, Daniela Kühn, Allan Lo, Deryk Osthus [1,2]

School of Mathematics
University of Birmingham
Birmingham B15 2TT, United Kingdom

Abstract

A fundamental theorem of Wilson states that, for every graph F, every sufficiently large F-divisible clique has an F-decomposition. Here a graph G is F-divisible if $e(F)$ divides $e(G)$ and the greatest common divisor of the degrees of F divides the greatest common divisor of the degrees of G, and G has an F-decomposition if the edges of G can be covered by edge-disjoint copies of F. We extend this result to graphs which are allowed to be far from complete: our results imply that every sufficiently large F-divisible graph G on n vertices with minimum degree at least $(1 - 1/(16|F|^4) + \varepsilon)n$ has an F-decomposition. Moreover, every sufficiently large K_3-divisible graph of minimum degree at least $0.956n$ has a K_3-decomposition. Our result significantly improves previous results towards the long-standing conjecture of Nash-Williams that every sufficiently large K_3-divisible graph with minimum degree $3n/4$ has a K_3-decomposition. For certain graphs, we can strengthen the general bound above. In particular, we obtain the asymptotically correct thresholds of $2n/3 + o(n)$ for C_4 and $n/2 + o(n)$ for even cycles of length at least 6. Our main contribution is a general method which turns an approximate decomposition into an exact one.

Keywords: minimum degree, edge decomposition

1 Introduction

Given a graph F, a graph G has an *F-decomposition* (is *F-decomposable*), if the edges of G can be covered by edge-disjoint copies of F. In this paper, we always consider decomposing a large graph G into edge-disjoint copies of some small fixed graph F. The first such result was given by Kirkman [7] in 1847, who proved that the complete graph K_n has a K_3-decomposition if and only if $n \equiv 1, 3 \mod 6$. To see that $n \equiv 1, 3 \mod 6$ is a necessary condition, note that if G has a K_3-decomposition, then the degree of each vertex of G is even and $e(G)$ is divisible by 3.

There are similar necessary conditions for the existence of an F-decomposition. For a graph G, let $\gcd(G)$ be the largest integer dividing the degree of every vertex of G. Given a graph F, we say that G is *F-divisible* if $e(G)$ is divisible by $e(F)$ and $\gcd(G)$ is divisible by $\gcd(F)$. Being F-divisible is a necessary condition for being F-decomposable. However, it is not sufficient: for example, C_6 does not have a K_3-decomposition. In this terminology, Kirkman proved that every K_3-divisible clique has a K_3-decomposition. The analogue of this for general graphs F instead of K_3 was an open problem for a century until it was solved by Wilson [12] in 1975. Wilson proved that, for every graph F, there exist an integer $n_0 = n_0(F)$ such that every F-divisible K_n with $n \geq n_0$ has an F-decomposition.

1.1 Decompositions of non-complete graphs

In contrast, it is well known that the problem of deciding whether a general graph G has an F-decomposition is NP-complete for every graph F that contains a connected component with at least three edges [2]. So a major question has been to determine the smallest minimum degree that guarantees an F-decomposition in any sufficiently large F-divisible graph G. Gustavsson [4] showed that, for every fixed graph F, there exists $\epsilon = \epsilon(F) > 0$ and $n_0 = n_0(F)$ such that every F-divisible graph G on $n \geq n_0$ vertices with minimum degree $\delta(G) \geq (1 - \epsilon)n$ has an F-decomposition. (This proof has not been without criticism.) In a recent breakthrough, Keevash [6] proved a hypergraph generalisation of Gustavsson's theorem. His result actually states that every sufficiently large dense quasirandom hypergraph has a decomposi-

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[2] Email: b.a.barber@bham.ac.uk, d.kuhn@bham.ac.uk, s.a.lo@bham.ac.uk, d.osthus@bham.ac.uk

tion into cliques (subject to the necessary divisibility conditions). The special case for complete hypergraphs settles a question regarding the existence of designs going back to the 19th century. Yuster [13] determined the asymptotic minimum degree threshold which guarantees an F-decomposition in the case when F is a bipartite graph with $\delta(F) = 1$ (which includes trees). For a survey regarding F-decomposition of hypergraphs, directed graphs and oriented graphs, we recommend [14].

Here, we substantially improve existing results when F is an arbitrary graph. For $F = K_3$, Nash-Williams [10] conjectured that every sufficiently large K_3-divisible graph G on n vertices with $\delta(G) \geq 3n/4$ has a K_3-decomposition. This conjecture is still wide open. For a general K_{r+1}, the following (folklore) conjecture is a natural extension of Nash-Williams's.

Conjecture 1.1 *For every $r \in \mathbb{N}$ with $r \geq 2$, there exists an $n_0 = n_0(r)$ such that every K_{r+1}-divisible graph G on $n \geq n_0$ vertices with $\delta(G) \geq (1 - 1/(r+2))n$ has a K_{r+1}-decomposition.*

The following result gives the first significant step towards the bound given by the above constructions and extends to decompositions into arbitrary graphs.

Theorem 1.2 *Let F be a graph and let $t := \max\{16\chi(F)^2(\chi(F)-1)^2, 6e(F)\}$. Then for each $\epsilon > 0$, there is an $n_0 = n_0(\epsilon, F)$ such that every F-divisible graph G on $n \geq n_0$ vertices with $\delta(G) \geq (1-1/t+\varepsilon)n$ has an F-decomposition.*

Note that, for any F, we have $t \leq 16|F|^4$. The best previous bound in this direction is the one given by Gustavsson [4], who claimed that, if F is complete, then a minimum degree bound of $(1-10^{-37}|F|^{-94})n$ suffices. For the special case of triangles we obtain the following improvement to Theorem 1.2.

Theorem 1.3 *There is an n_0 such that every K_3-divisible graph G on $n \geq n_0$ vertices with $\delta(G) \geq 0.956n$ has a K_3-decomposition.*

1.2 Approximate F-decompositions

Our main contribution is actually a result that turns an 'approximate' F-decomposition into an exact F-decomposition. Let G be a graph on n vertices. For a graph F and $\eta \geq 0$, an *η-approximate F-decomposition* $\mathcal{F}$ of G is a set of edge-disjoint copies of F covering all but at most ηn^2 edges of G. Note that a 0-approximate F-decomposition is an F-decomposition. For $n \in \mathbb{N}$, let $\delta_F^\eta(n)$ be the smallest constant δ such that every graph G on n vertices with $\delta(G) \geq \delta n$ has a η-approximate F-decomposition.Let $\delta_F^\eta := \limsup_{n\to\infty} \delta_F^\eta(n)$

be the *η-approximate F-decomposition threshold*. Clearly $\delta_F^{\eta_1} \geq \delta_F^{\eta_2}$ for all $\eta_1 \leq \eta_2$. Note that there are graphs with $\lim_{\eta\to 0} \delta_F^\eta = \delta_F^0$, and graphs for which this equality does not hold.

Our main result relates the 'decomposition threshold' to the 'approximate decomposition threshold' and an additional minimum degree condition for r-regular graphs F. The dependence on r gives the correct order of magnitude.

Theorem 1.4 *Let F be an r-regular graph. Then for each $\epsilon > 0$, there exists an $n_0 = n_0(\epsilon, F)$ and an $\eta = \eta(\varepsilon, F)$ such that every F-divisible graph G on $n \geq n_0$ vertices with $\delta(G) \geq (\delta + \epsilon)n$, where $\delta := \max\{\delta_F^\eta, 1 - 1/3r\}$, has an F-decomposition.*

Our proof of Theorem 1.4 can be applied to give better bounds for some specific choices of F. For example, we prove the following result on cycle decompositions.

Theorem 1.5 *Let $\ell \in \mathbb{N}$ with $\ell \geq 3$, and let $\delta_4 := 1/2$; $\delta_\ell := 2/3$ if $\ell \geq 6$ is even; and $\delta_\ell := 0.956$ if ℓ is odd. Then for each $\epsilon > 0$, there is an $n_0 = n_0(\epsilon, \ell)$ such that every C_ℓ-divisible graph G on $n \geq n_0$ vertices with $\delta(G) \geq (\delta_\ell + \epsilon)n$ has a C_ℓ-decomposition.*

The special case when $\ell = 4$ improves a result of Bryant and Cavenagh [1]. For even cycles the value of the constant δ_ℓ in Theorem 1.5 is the best possible.

2 Sketches of proofs

2.1 Proof of Theorem 1.2 using Theorem 1.4.

The idea of this proof is quite natural. Given a graph F as in Theorem 1.2, we find an F-decomposable regular graph R such that both the degree r of R and the η-approximate decomposition threshold δ_R^η are not too large. By removing a small number of copies of F from G, we may assume that G is also R-divisible. By Theorem 1.4, G has an R-decomposition and so an F-decomposition, provided $\delta(G) \geq \max\{\delta_R^\eta, 1 - 1/3r\}$. To obtain the explicit bound on $\delta(G)$, we apply a result of Dukes [3] on fractional decompositions in graphs of large minimum degree together with a result of Haxell and Rödl [5] relating fractional decompositions to approximate decompositions.

2.2 Proof of Theorem 1.4.

The proof of Theorem 1.4 uses the 'absorbing' approach. This method was first used for finding K_3-factors (that is, a spanning union of vertex-disjoint copies of K_3) by Krivelevich [8] and for finding Hamilton cycles in hypergraphs by Rödl, Ruciński and Szemerédi [11]. An absorbing approach for finding decompositions was first used by Kühn and Osthus [9].

More precisely, the basic idea behind the proof of Theorem 1.4 can be described as follows. Let G be a graph as in Theorem 1.4. Suppose that we can find a sparse F-divisible subgraph A^* of G which is an F-absorber in the following sense: $A^* \cup H^*$ has an F-decomposition whenever H^* is a sparse F-divisible graph on $V(G)$ which is edge-disjoint from A^*. Let G' be the subgraph of G remaining after removing the edges of A^*. Since A^* is sparse, $\delta(G') \geq (\delta_F^\eta + \varepsilon/2)n$. By the definition of δ_F^η, G' has an η-approximate F-decomposition $\mathcal{F}$. Let $H^* := G' - \bigcup \mathcal{F}$ be the leftover. Note that H^* is also F-divisible. Since $A^* \cup H^*$ has an F-decomposition, so does G.

Unfortunately, this naïve approach fails for the following reason: we have no control on the leftover H^*. More precisely, the natural way to obtain A^* would be to construct it as the edge-disjoint union of graphs A such that each such A has an F-decomposition and, for each possible leftover graph H^*, there is a distinct A so that $A \cup H^*$ has an F-decomposition. However, even if $H^* = C_6$, the number of possibilities for H^* is at least $\binom{n}{6}$. So we have no hope of finding all the required graphs A in G (and thus to construct A^*). To overcome this problem, we reduce the number of possible configurations of H^* (in turn reducing the number of graphs A required) as follows. Roughly speaking, we iteratively find approximate decompositions of the leftover so that eventually our final leftover H^* only has $O(n)$ edges whose location is very constrained—so one can view this step as finding a 'near optimal' F-decomposition.

To illustrate this, suppose that $m \in \mathbb{N}$ is bounded and n is divisible by m. Let $\mathcal{P} := \{V_1, \dots, V_q\}$ be a partition of $V(G)$ into parts of size m (so $q = n/m$). We further suppose that H^* is a vertex-disjoint union of F-divisible graphs $H_1^*, \dots, H_q^*$ such that $V(H_i^*) \subseteq V_i$ for each i. Hence to construct A^*, we only need to find one A for each possible H_i^*. For a fixed i, there are at most $2^{\binom{|V_i|}{2}} = 2^{\binom{m}{2}}$ possible configurations of H_i^*. Since m is bounded, in order to construct A^* we would only need to find $q2^{\binom{m}{2}} = 2^{\binom{m}{2}}n/m$ different A.

We now describe in more detail the iterative approach which achieves the above setting. Recall that G' is the subgraph of G remaining after removing all the edges of A^*. Since A^* is sparse, G' has roughly the same properties as G. Our new objective is to find edge-disjoint copies of F covering all edges of G' that do not lie entirely within V_i for some i. Since each V_i has bounded size, these edge-disjoint copies of F will cover all but at most a linear number of edges of G'. As indicated above, we use an iterative approach to achieve this. We proceed as follows. Let $k \in \mathbb{N}$. Let $\mathcal{P}_1$ be an equipartition of $V(G)$ into k parts, and let G_1 be the k-partite subgraph of G' induced by $\mathcal{P}_1$ (here k is large but bounded). Suppose that we can cover the edges of G_1 by copies of F which use only a small proportion of the edges not in G_1. Call the leftover graph H_1. Let $\mathcal{P}_2$ be an equipartition of $V(G)$ into k^2 parts obtained by dividing each $V \in \mathcal{P}_1$ into k parts. Let G_2 be the k^2-partite subgraph of H_1 induced by $\mathcal{P}_2$. Each component of G_2 will form a k-partite graph lying within some $V \in \mathcal{P}_1$. So by applying the same argument to each component of G_2 in turn and iterating $\log_k(n/m)$ times we obtain an equipartition $\mathcal{P} = \mathcal{P}_\ell$ of $V(G)$ with $|V| = m$ for each $V \in \mathcal{P}$ such that all edges of G' that do not lie entirely within some $V \in \mathcal{P}$ can be covered by edge-disjoint copies of F.

References

[1] D. Bryant and N. Cavenagh, *Decomposing graphs of high minimum degree into 4-cycles*, J. Graph Theory, **79** (2015), 167–177.

[2] D. Dor and M. Tarsi, *Graph decomposition is NPC - a complete proof of Holyer's conjecture*, STOC '92 Proceedings of the twenty-fourth annual ACM symposium on Theory of computing, (1992), 252–263.

[3] P. Dukes, *Rational decomposition of dense hypergraphs and some related eigenvalue estimates*, Linear Algebra Appl., **436** (2012), 3736–3746, (see arXiv:1108.1576 for an erratum).

[4] T. Gustavsson, *Decompositions of large graphs and digraphs with high minimum degree*, PhD thesis, Univ. of Stockholm, 1991.

[5] P.E. Haxell and V. Rödl, *Integer and fractional packings in dense graphs*, Combinatorica, **21** (2001), 13–38.

[6] P. Keevash, *The existence of designs*, arXiv preprint arXiv:1401.3665.

[7] T.P. Kirkman, *On a problem in combinatorics*, Cambridge Dublin Mathematical Journal, **2** (1847) 191–204.

[8] M. Krivelevich, *Triangle factors in random graphs*, Combin. Probab. Comput., **6** (1997), 337–347.

[9] D. Kühn and D. Osthus, *Hamilton decompositions of regular expanders: a proof of Kelly's conjecture for large tournaments*, Adv. Math., **237** (2013), 62–146.

[10] C.St.J.A. Nash-Williams, *An unsolved problem concerning decomposition of graphs into triangles*, Combinatorial Theory and its Applications III, North Holland (1970), 1179–183.

[11] V. Rödl, A. Ruciński, and E. Szemerédi, *A Dirac-type theorem for 3-uniform hypergraphs*, Combin. Probab. Comput., **15** (2006), 229–251.

[12] R.M. Wilson, *Decompositions of complete graphs into subgraphs isomorphic to a given graph*, Proceedings of the Fifth British Combinatorial Conference (Univ. Aberdeen, Aberdeen (1975), 647–659.

[13] R. Yuster, *The decomposition threshold for bipartite graphs with minimum degree one*, Random Structures Algorithms, **21** (2002), 121–134.

[14] R. Yuster, *Combinatorial and computational aspects of graph packing and graph decomposition*, Computer Science Review, **1** (2007), 12–26.

[15] R. Yuster, *H-packing of k-chromatic graphs*, Mosc. J. Comb. Number Theory, **2** (2012), 73–88.