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Some Remarks on Rainbow Connectivity

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Abstract

An edge- (vertex-) coloured graph is rainbow connected if there is a rainbow path between any two vertices, i.e. a path all of whose edges (internal vertices) carry distinct colours. Rainbow edge (vertex) connectivity of a graph G is the smallest number of colours needed for a rainbow edge (vertex) colouring of G. In this paper we propose a very simple approach to studying rainbow connectivity in graphs. Using this idea, we give a unified proof of several new and known results, focusing on random regular graphs.

Keywords: Rainbow connectivity, random regular graph, diameter

1 Introduction

An edge-coloured graph G is called *rainbow connected* if there is a rainbow path between any two vertices, that is a path on which all edges have distinct colours. Any connected graph G of order n can be made rainbow-connected using $n-1$ colours by choosing a spanning tree and giving each edge of the spanning tree a different colour. Hence we can define emphrainbow connectivity, $rc(G)$, as the minimal number of colours needed to ensure that G is rainbow connected.

Rainbow connectivity is introduced in 2008 by Chartrand et al. [3] as a way of strengthening the notion of connectivity. The concept has attracted a considerable amount of attention in recent years, see for example [1], [2], [4], [6], and the survey [7]. It is also of interest in applied settings, such as securing sensitive information transfer and networking (see, e.g., [4]).

We are interested in upper bounds for rainbow connectivity, first studied by Caro et al. [1]. The trivial lower bound is $rc(G) \geq diam(G)$, and it turns out that for many classes of graphs, this is a reasonable guess for the value of rainbow connectivity. Krivelevich and Yuster [6] showed that a connected n-vertex graph G of minimum degree δ satisfies $rc(G) \leq \frac{20n}{\delta}$, which is of the same order as the elementary bound $diam(G) \leq \frac{3n}{\delta+1}$ proved by Erdős et al. in [5]. Then Chandran et al. [2] settled this question by proving $rc(G) \leq \frac{3n}{\delta+1}+3$, which is asymptotically tight.

A random r-regular graph of order n is a graph sampled from $G_{n,r}$, which denotes the uniform probability space of all r-regular graphs on n labelled vertices. These graphs were extensively studied in the last 30 years, see, e.g., [8]. In this paper we consider $G_{n,r}$ for r constant and $n \to \infty$. We say that an event holds *with high probability* (whp) if its probability tends to 1 as n tends to infinity, but only over the values of n for which nr is even (so that $G_{n,r}$ is non-empty).

A random r-regular graph has strong connectivity properties, for example, the diameter of $G_{n,r}$ is whp asymptotic to $\frac{\log n}{\log(r-1)}$. Dudek et al. [4] showed that $rc\left(G_{n,r}\right) = O(\log n)$ whp, which is the correct dependence on n. We will

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return to this result later.

The aim of this note is to present a simple approach which immediately implies results on rainbow colouring of several classes of graphs. It provides a unified approach to various settings, yields new theorems, strengthens some of the earlier results and simplifies the proofs. It is based on edge- and vertex-splitting.

The main idea of the edge-splitting lemma is simple: we decompose G into two edge-disjoint spanning trees T_1 and T_2 with a common root vertex and small diameters. We use different palettes for edges of T_1 and T_2, ensuring that each tree contains a rainbow path from any vertex to the root. Hence if we can get the diameters of T_1 and T_2 'close' to the diameter of G (say within a constant factor), then we have obtained a strong result.

We exhibit a few applications of the lemma. First we use it to give a straightforward proof of the result of Krivelevich and Yuster [6], that is

Theorem 1.1 *For a connected n-vertex graph G of minimum degree $\delta \geq 4$, $rc(G) \leq \dfrac{16n}{\delta}$.*

Next we turn to random regular graphs.The rainbow colouring of $G_{n,r}$ of Dudek et al. [4] typically uses $\Omega(r \log n)$ colours, which for large r is significantly bigger than the diameter of $G_{n,r}$. Using our splitting lemma we can improve it to an asymptotically tight bound.

Theorem 1.2 *There is an absolute constant c such that for $r \geq 5$, $rc(G_{n,r}) \leq \dfrac{c \log n}{\log r}$ whp.*

For $r \geq 6$, the theorem is an immediate consequence of the contiguity of different models of random regular graphs. With little extra work, our approach also works for 5-regular graphs.

The question of which characteristics of $G_{n,r}$ ensure small rainbow connectivity arises naturally. Recalling that expander graphs also have diameter logarithmic in n, it makes sense to look at expansion properties. The following theorem, proved in the full paper, generalises the previous result on $G_{n,r}$.

Theorem 1.3 *Let $\epsilon > 0$. Let G be a graph of order n and degree r whose edge expansion is at least ϵr. Furthermore, assume $r \geq 64\epsilon^{-1} \log (64\epsilon^{-1})$. Then $rc(G) = O(\epsilon^{-1} \log n)$.*

In particular, this theorem applies to (n, r, λ)-graphs with $\lambda \leq r(1 - 2\epsilon)$, i.e. n-vertex r-regular graphs whose all eigenvalues except the largest one are at most λ in absolute value.

Krivelevich and Yuster [6] have introduced the corresponding concept of *rainbow vertex connectivity* $rvc(G)$, the minimal number of colours needed for a rainbow colouring of vertices of G. The only point to clarify is that a path is said to be rainbow if its *internal* vertices carry distinct colours. The easy bounds $diam(G) - 1 \leq rvc(G) \leq n$ also hold in this setting. Krivelevich and Yuster have demonstrated that it is impossible to bound the rainbow connectivity of G in terms of its vertex rainbow connectivity, or the other way around. They also bound $rvc(G)$ in terms of the minimal degree.

Our approach essentially works for vertex colouring as well. In Section 3 we present the vertex-splitting lemma. It is then used to prove the vertex-colouring analogue of Theorem 1.2 on random regular graphs.

Theorem 1.4 *There is an absolute constant c such that whp $rvc(G_{n,r}) \leq \frac{c \log n}{\log r}$ for all $r \geq 26$.*

2 Edge rainbow connectivity

We now state and prove the splitting lemma.

Lemma 2.1 *Let $G = (V, E)$ be a graph. Suppose there are two connected spanning subgraphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ such that $|E_1 \cap E_2| \leq c$. Then $rc(G) \leq diam(G_1) + diam(G_2) + c$.*

Proof. Let $B = E_1 \cap E_2$. Colour the edges of B in distinct colours. These colours will remain unchanged, and the remaining edges get coloured according to graph distances in G_1 and G_2, denoted by d_1 and d_2. Choose an arbitrary $v \in V$ and define distance sets $U_j = \{u \in V : d_1(v, u) = j\}$ and $W_j = \{u \in V : d_2(v, u) = j\}$. For $1 \leq j \leq diam(G_1)$, colour the edges between U_{j-1} and U_j with colour a_j. Similarly, using a new palette (b_j), colour the edges between W_{j-1} and W_j with colour b_j for each $1 \leq j \leq diam(G_2)$. The colouring indeed uses at most $diam(G_1) + diam(G_2) + c$ colours.

To see that it is a rainbow colouring, look at two vertices x_1 and x_2 in V. A shortest path P_i from x_i to v is rainbow for our choice of colouring. If P_1 and P_2 are edge-disjoint, the concatenation is a rainbow path between x_1 and x_2. Otherwise, if P_1 and P_2 intersect in one of the edges of B, we walk from x_1 along P_1 to the earliest common edge. We use this edge to switch to P_2 and walk to x_2.

□

2.1 Rainbow connectivity and minimum degree

In this setting, the best possible result has been shown by Chandran et al [2]. Namely, a connected graph G of order n and minimum degree δ satisfies $rc(G) \leq \frac{3n}{\delta+1} + 3$. We show how the splitting lemma can be used with basic graph theory to obtain a good upper bound, $rc(G) \leq \frac{16n}{\delta}$.

Proof. [Sketch proof of Theorem 1.1] We split the graph $G = (V, E)$ into two spanning subgraphs of minimum degree at least $\frac{\delta-1}{2}$. Assume that all vertices of G have even degree, since this can be ensured by adding a matching to G. Then, using connectedness of G, order its edges along an Eulerian cycle $e_1, e_2 \ldots e_m$, and define
$F_1 = \{e_j : j \in [m] \text{ even}\} \quad \text{and} \quad F_2 = \{e_j : j \in [m] \text{ odd}\}$.

The graph formed by F_1 may not be connected. But since the minimum degree of this graph is $\frac{\delta-1}{2}$, each connected component has order at least $\frac{\delta}{2}$. Hence the number of components of F_1 is at most $\frac{2n}{\delta}$, so we can add a set $B_1 \subset E$ such that $G_1 = (V, F_1 \cup B_1)$ is connected, and $|B_1| \leq \frac{2n}{\delta}$. We define the set B_2 analogously.

An elementary graph-theoretic argument (see [5]) shows that subgraphs G_1 and G_2 of G have diameters at most $\frac{3n}{\delta/2+1} \leq \frac{6n}{\delta}$. Applying the edge-splitting lemma to G_1 and G_2 gives $rc(G) \leq \frac{6n}{\delta} + \frac{6n}{\delta} + \frac{4n}{\delta} \leq \frac{16n}{\delta}$. □

2.2 Random regular graphs

Two sequences of probability spaces $\mathcal{F}_n$ and $\mathcal{G}_n$ on the same underlying measurable spaces are called *contiguous*, written $\mathcal{F}_n \approx \mathcal{G}_n$, if a sequence of events (A_n) occurs whp in $\mathcal{F}_n$ if and only if it occurs whp in $\mathcal{G}_n$. Let $\mathcal{G}$ and $\mathcal{G}'$ be two models of random graphs on the same vertex set. We get a new random graph G by taking the union of independently chosen graphs $G_1 \in \mathcal{G}$ and $G_2 \in \mathcal{G}'$, conditional on the event $E(G_1) \cap E(G_2) = \emptyset$. The probability space of such disjoint unions is denoted by $\mathcal{G} \oplus \mathcal{G}'$.

It is known that $G_{n,r}$ is contiguous with any other model which builds an r-regular graph as an edge-disjoint union of random regular graphs and Hamiltonian cycles (see, e.g., [8]). The specific results we use in proving Theorem 1.2 are $G_{n,r+r'} \approx G_{n,r} \oplus G_{n,r'}$ and $G_{n,r+2} \approx G_{n,r} \oplus H_n$, where H_n is a random Hamiltonian cycle on $[n]$. Recall that Theorem 1.2 says that for $r \geq 5$, $rc(G_{n,r}) \leq \frac{c \log n}{\log r}$ whp.

Proof. [Proof of Theorem 1.2 for $r \geq 6$.] As usually, we assume that rn is even, and define r_i so that G_{n,r_i} are non-empty for $i = 1, 2$. If r is odd, then n

is even and we can set $r_i = \frac{r\pm1}{2}$. Otherwise, we set $r_1 = r_2 = \frac{r}{2}$ or $r_i = \frac{r}{2} \pm 1$ as appropriate. The observation at the end of the proof resolves the case $r = 6$.

Let G_i be a random r_i-regular graph. Then with high probability $diam(G_i) \leq \frac{(1+o(1))\log n}{\log(r_i-1)} \leq \frac{c\log n}{2\log r}$, where c is a suitable constant. Let G be the union of two such edge-disjoint graphs G_1 and G_2. The splitting lemma gives $rc(G) \leq \frac{c\log n}{\log r}$.

Since G is a random element of $G_{n,r_1} \oplus G_{n,r_2}$, the random r-regular graph has the same property whp.

For $r = 6$, we model $G_{n,6}$ as the disjoint union of two nearly 3-regular graphs using $G_{n,6} \approx H_n \oplus H_n \oplus H_n$. □

Our approach also works for $r = 5$, but we omit the proof in this note. Since $G_{n,5} \approx G_{n,1} \oplus H_n \oplus H_n$, we can model our 5-regular graph as a union of two random graphs G_1 and G_2, where each G_i is an edge-disjoint union of a Hamiltonian cycle and a random matching of size $\lfloor \frac{n}{4} \rfloor$.

In the full paper, we show that G_i has diameter $O(\log n)$ whp. The key observation is that G_i can be built in two steps as follows. Denote $m = \lfloor \frac{n}{4} \rfloor$. First we select a random subset $B = \{b_1, b_2, \ldots, b_{2m}\} \subset [n]$ of order $2m$, and then independently a random perfect matching on $\{b_1, b_2, \ldots, b_{2m}\}$.

3 Vertex rainbow connectivity

We now state the vertex-colouring analogue of Lemma 2.1. The proof follows the same steps.

Lemma 3.1 *Let $G = (V, E)$ be a graph. Suppose that $V_1, V_2 \subset V$ satisfy: 1) $V_1 \cup V_2 = V$; 2) $|V_1 \cap V_2| \leq c$; 3) every vertex $v \in V_1$ has a neighbour in V_2 and vice versa; 4) $G[V_i]$ is connected, for $i = 1, 2$. Then*

$$rvc(G) \leq diam(G[V_1]) + diam(G[V_2]) + c + 2.$$

3.1 Random regular graphs

We will split the vertices of G using the following lemma. The proof is a standard application of the Lovász Local Lemma. If we are only interested in large values of r, we may take γ close to 0.5.

Lemma 3.2 *Let $r \geq 26$. Then there is a constant $\gamma \geq 0.12$ such that the vertices of any r-regular graph G can be partitioned as $V = V_1 \cup V_2$, and each $v \in V_i$ satisfies $r > \deg_{G[V_i]}(v) \geq \gamma r$.*

To use such a partition, we need an estimate on the number of edges spanned by subsets of vertices of $G_{n,r}$. Since r is constant with n, $G_{n,r}$ is contiguous to the configuration (or pairing) model of random regular graphs, described for example in [8]. Via the configuration model, we prove the following lemma.

Lemma 3.3 *Fix a natural number $r \geq 3$, and let $G = G_{n,r}$.*

(i) *For γ' satisfying $\gamma' r \geq 3$, there is a constant $\alpha = \alpha(\gamma') > 0$ such that whp all sets $S \subset [n]$ of vertices of G of order up to αn span fewer than $\frac{|S|\gamma' r}{2}$ edges.*

(ii) *There is an absolute constant $\beta > 0$ such that whp all sets $S \subset [n]$ of vertices of G of order up to $\frac{\beta n}{r}$ span fewer than $3|S|$ edges.*

We can now prove the main result of this section, $rvc(G_{n,r}) = O\left(\frac{\log n}{\log r}\right)$ whp for $r \geq 26$.

Proof. [Proof of Theorem 1.4.] Let G be a random r-regular graph, $\gamma = 0.12$. Use Lemma 3.2 to obtain a partition $V = U_1 \cup U_2$ such that $r > \deg_{G[U_i]}(v) \geq \gamma r$ for all $v \in U_i$ and $i = 1, 2$.

All statements about G from now on will hold with high probability. In particular, we assume that G satisfies Lemma 3.3 with $\gamma' = \frac{\gamma}{1+\epsilon}$, where $\epsilon > 0$ is chosen so that $\frac{\gamma r}{1+\epsilon} > 3$. (e.g. $\epsilon = 0.03$ is small enough). We only need the extra $(1+\epsilon)^{-1}$ factor later, for Claim 3. Such edge distribution implies that each connected component of $G[U_i]$ contains at least αn vertices, where α is the constant from Lemma 3.3.

Claim 1. We can find $W_i \subset V$ such that $W_i = O(1)$ and $G[U_i \cup W_i]$ is connected.

For a set of vertices $A \subset V$, denote $\Gamma^j(A) = \{v \in V : d_G(v, A) \leq j\}$. It is well-known that a random regular graph has good expansion properties, i.e. there is a constant $\phi > 0$ such that whp $|\Gamma(A)| \geq (1+\phi)|A|$ whenever $|A| \leq \frac{n}{2}$. Now suppose that A has linear order, $|A| \geq \alpha n$, and take an integer $l > \frac{\log \alpha^{-1} - \log 2}{\log(1+\phi)}$. Iterating the expansion property gives that $|\Gamma^l(A)| > \frac{n}{2}$. To prove Claim 1, suppose A and B are vertex sets of two connected components of $G[U_i]$, each of order at least αn. We just showed that $\Gamma^l(A) \cap \Gamma^l(B) \neq \emptyset$, so there is a path of length at most $2l$ from A to B in G. Adding the vertices of this path to W_i reduces the number of connected components by one, so repeating this step α^{-1} times ensures that $V_i = U_i \cup W_i$ spans a connected graph $G_i = G[V_i]$. The vertex sets V_1 and V_2 now satisfy $|V_1 \cap V_2| = O(1)$.

Claim 2. For $r \geq 104$ (so that $\gamma r \geq 12$), every $T \subset V_i$ of order at most $\frac{\beta n}{\gamma r^2}$ satisfies $|\Gamma_{G_i}(T)| \geq \left(1 + \frac{\gamma r}{12}\right)|T|$.

Suppose T does not satisfy the claim, and let $S = \Gamma_{G_i}(T)$. Note that by the hypothesis $|S| \le \left(1+\frac{\gamma r}{12}\right) \cdot \frac{\beta n}{\gamma r^2} < \frac{\beta n}{r}$. Hence we can deduce from Lemma 3.3 (ii) that S spans fewer than $3|S|$ edges, which contradicts the minimum degree of vertices inside T.

Claim 3. Let α be the constant from Lemma 3.3 (i) and $\epsilon > 0$ as above. Every subset $T \subset V_i$ of order at most $\frac{\alpha n}{1+\epsilon}$ satisfies $|\Gamma_{G_i}(T)| \ge (1+\epsilon)|T|$.
The proof is analogous to Claim 2.

For $r \ge 104$, Claim 2 implies that starting from any vertex $v \in V_i$, we can expand in G_i to a set of order $\frac{\beta n}{\gamma r^2}$ in $\frac{c_1 \log n}{\log r}$ steps, where c_1 is a constant independent of r and n. Further $O(\log r)$ steps give a set of order $\frac{\alpha n}{1+\epsilon}$, by Claim 3. For finitely many values $r < 104$, we skip the first expansion and readjust the constants.

Denote $k = \frac{c \log n}{\log r}$, where $c > c_1$ is sufficiently large for the previous argument to go through. Suppose the diameter of G_i is larger than $\frac{4k}{\alpha}$, and take x_0 and x_R such that the shortest path $x_0 x_1 \dots x_R$ is longer than $\frac{4k}{\alpha}$. Then we can use the procedure above to expand from vertices x_0, x_{3k}, $x_{6k} \dots$ by k steps to get $\frac{4}{3\alpha}$ *disjoint* (by the choice of the path) neighbourhoods, each of order $\frac{\alpha n}{1+\epsilon}$, which is a contradiction. Thus applying Lemma 3.1 to subsets V_1 and V_2 gives $rvc(G) \le \frac{9c \log n}{\alpha \log r}$, as required. □

Concluding remarks

In this note we proposed a simple approach to studying rainbow connectivity and rainbow vertex connectivity in graphs. Using it we gave a unified proof of several known results, as well as of some new ones. Two obvious interesting questions which remain open are to show that rainbow edge connectivity and rainbow vertex connectivity of random 3-regular graphs on n vertices are logarithmic in n.

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