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Partitioning H-minor free graphs into three subgraphs with no large components

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Abstract

We prove that for every graph H, if a graph G has no H minor, then $V(G)$ can be partitioned into three sets such that the subgraph induced on each set has no component of size larger than a function of H and the maximum degree of G. This answers a question of Esperet and Joret and improves a result of Alon, Ding, Oporowski and Vertigan and a result of Esperet and Joret. As a corollary, for every positive integer t, if a graph G has no K_{t+1} minor, then $V(G)$ can be partitioned into $3t$ sets such that the subgraph induced on each set has no component of size larger than a function of t. This corollary improves a result of Wood.

Keywords: Graph minors, coloring, partitioning.

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1 Introduction

For a graph G and a set X of vertices, we write $G[X]$ to denote its subgraph induced on X. The famous Four Color Theorem states that every planar graph G admits a partition of its vertex set into four sets X_1, X_2, X_3, X_4 such that for $1 \leq i \leq 4$, every component of $G[X_i]$ has at most one vertex. Certainly there are planar graphs whose vertex set cannot be partitioned into three such sets. However, Esperet and Joret [2] proved that the number of sets can be reduced to three, if we relax each X_i to induce a subgraph having no component of size larger than a function of the maximum degree of G. More generally, they proved that for every surface Σ and for every positive integer Δ, there exists an integer N such that every graph G of maximum degree at most Δ can be partitioned into three sets X_1, X_2, X_3 such that for $1 \leq i \leq 3$, every component of $G[X_i]$ has at most N vertices. The number "three" in their theorem is best possible, since a large triangular grid has maximum degree six but its vertex set cannot be partitioned into two sets such that each set induces a subgraph with no component of small size by the famous HEX lemma.

On the other hand, Alon, Ding, Oporowski and Vertigan [1] proved that graphs in a broader class admit such a partition into four parts. We say that graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. Alon et al. [1] proved that for every graph H and for every positive integer Δ, there exists an integer N such that if H is not a minor of a graph G of the maximum degree at most Δ, then $V(G)$ can be partitioned into four sets X_1, X_2, X_3, X_4 such that for $1 \leq i \leq 4$, every component of $G[X_i]$ has at most N vertices.

In this paper, we provide a positive answer of a question of Esperet and Joret [2, Question 5.1] by proving the following strengthening of the mentioned theorem of Esperet and Joret and the mentioned theorem of Alon et al.

Theorem 1.1 *For every graph H and every positive integer Δ, there exists an integer N such that if H is not a minor of a graph G of maximum degree at most Δ, then $V(G)$ can be partitioned into three sets X_1, X_2, X_3 such that for $1 \leq i \leq 3$, every component of $G[X_i]$ has at most N vertices.*

As an application of Theorem 1.1, we investigate the following relaxation of Hadwiger's conjecture: what is the minimum k as a function of t such that for some N, every graph G with no K_{t+1} minor admits a partition of $V(G)$ into k sets $X_1, X_2, \ldots, X_k$ with the property that each $G[X_i]$ has no component on more than N vertices? Hadwiger's conjecture [4], if true, would imply that

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$k = t$. Kawarabayashi and Mohar [5] proved that $k \leq \lceil 15.5(t+1) \rceil$, and Wood [7] proved that $k \leq \lceil 3.5t + 2 \rceil$. Theorem 1.1 leads to the following improvement of these results by using a recent theorem of Edwards, Kang, Kim, Oum, and Seymour [3].

Theorem 1.2 *For every positive integer t, there exists N such that if K_{t+1} is not a minor of a graph G, then $V(G)$ can be partitioned into $3t$ sets $X_1, X_2, \ldots, X_{3t}$ such that for $1 \leq i \leq 3t$, every component of $G[X_i]$ has at most N vertices.*

Now we prove Theorem 1.2. By Edwards et al. [3], there exists an integer s such that $V(G)$ can be partitioned into t sets $V_1, V_2, \ldots, V_t$ such that the maximum degree of $G[V_i]$ is at most s for $1 \leq i \leq t$. By Theorem 1.1, there exists an integer N depending only on t such that for $1 \leq i \leq t$, V_i can be partitioned into three sets V_{i1}, V_{i2}, V_{i3} and each of $G[V_{i1}]$, $G[V_{i2}]$, $G[V_{i3}]$ has no component having size larger than N vertices. Therefore, $\{V_{ij} : 1 \leq i \leq t, 1 \leq j \leq 3\}$ is a desired partition. This proves Theorem 1.2.

2 Sketch of the proof of Theorem 1.1

A *k-coloring* of a graph G is a function mapping the vertices of G into the set $\{1, 2, \ldots, k\}$. A *monochromatic component* is a component of the subgraph induced by the vertices of the same color in a given k-coloring. Note that the color classes in a k-coloring form a partition of $V(G)$. So our objective is to find a 3-coloring of G such that every monochromatic component has small size.

To prove Theorem 1.1, we in fact prove a stronger result in which we allow few vertices are precolored and show that the precoloring can be extended to a 3-coloring of the whole graph such that the size of the monochromatic components meeting precolored vertices are relatively small. Note that Theorem 1.1 is an immediate corollary of the following theorem by taking $Y = \emptyset$.

Theorem 2.1 *For every graph L and positive integer Δ, there exists an integer η such that if L is not a minor of a graph G of maximum degree at most Δ, then for every subset Y of $V(G)$ with $|Y| \leq \eta$, every 3-coloring of Y can be extended to that of G satisfying the following.*

(i) *The union of all monochromatic components of G meeting Y contains at most $|Y|^2\Delta$ vertices.*

(ii) *Every monochromatic component of G contains at most $\eta^2\Delta$ vertices.*

Our proof of Theorem 2.1 uses the machinery in the Graph Minors series of Robertson and Seymour. The following statements require several definitions to be formally stated, so we only include informal descriptions here. A theorem of Robertson and Seymour [6] states that for every graph G not containing a fixed graph as a minor and for every "highly connected subgraph" T in G, G can be "decomposed" into pieces such that the "root piece" contains T and is "almost embeddable" in a surface of bounded genus.

An essential step toward our proof of Theorem 2.1 is to prove a weaker version of Theorem 2.1 but generalize the result of Esperet and Joret [2] on graphs embeddable in a fixed surface to graphs that are "almost embeddable" in a fixed surface.

Lemma 2.2 *For every surface Σ and for every integers a, b, c, there exists a number N such that every "(a, b, c)-almost embeddable" graph G of maximum degree Δ and for every subset Y of $V(G)$, every 3-coloring of Y can be extended to that of G such that every monochromatic component has at most N vertices and the union of all monochromatic components meeting Y contains at most $N|Y|$ vertices.*

Now we are ready to sketch the proof of Theorem 2.1. We shall proceed by induction on the number of vertices of G. We say that a 3-coloring of G is *Y-good* if it satisfies the conclusions of Theorem 2.1.

First, we show that we may assume that $|Y| \geq \eta/\Delta^2$. Suppose that $|Y| < \eta/\Delta^2$. Let X be the union of Y and its neighbors, and let Z be the set of vertices of G not in X but with distant two from some vertex in Y. Note that $|Z| < \eta$. We apply induction to $G - X$ with every vertex in Z precolored by color 1 to obtain a Z-good 3-coloring of $G - X$. Then further coloring the neighbors of Y with color 2 leads to a Y-good 3-coloring of G, since the union of the monochromatic components meeting Y contains at most $|Y|(\Delta + 1)$ vertices.

Second, we show that G contains a "highly connected subgraph" that contains most of vertices of Y. A *separation* of a graph G is a pair of edge-disjoint subgraphs (A, B) such that $A \cup B = G$. The *order* of (A, B) is $|V(A) \cap V(B)|$. For every separation (A, B) of G, we define $Y_A = (Y \cap V(A)) \cup (V(A) \cap V(B))$ and $Y_B = (Y \cap V(B)) \cup (V(A) \cap V(B))$. Suppose that there exists a separation (A, B) of small order such that both A and B contain many vertices in Y, then $|Y_A|$ and $|Y_B|$ are less than η. We color $V(A) \cap V(B) - Y$ by color 1 and apply induction to A and B, respectively, to obtain a Y_A-good coloring c_A of A and a Y_B-good colorings c_B of B. Then we obtain a 3-coloring of G by combining the coloring c_A and c_B. Since every monochromatic component intersects both $A - V(B)$ and $B - V(A)$ must intersect $V(A) \cap V(B) \subseteq Y_A \cap V_B$, the size of the union of all such monochromatic components in G is at most the sum of the monochromatic components of A and B meeting Y_A and Y_B with respect to c_A and c_B, respectively. Then the superadditivity of quadratic functions shows that the 3-coloring of G is Y-good.

Therefore, there exists a "highly connected subgraph" T in G containing most of the vertices in Y. The structure theorem of Robertson and Seymour for excluding a fixed graph as a minor [6] implies that G can be "decomposed" into pieces such that the "root piece" contains T and is "almost embeddable" in a surface of bounded genus. Hence, Lemma 2.2 shows that the precoloring on $Y \cap V(T)$ can be extended to a 3-coloring c_T of T such that each monochromatic component contains at most $N|Y \cap V(T)|$ vertices. In particular, c_T is $(Y \cap V(T))$-good as $|Y \cap V(T)|$ is large.

Finally, since $|Y \cap W|$ is small for each "non-root piece" W, the induction hypothesis ensures that the 3-coloring on $V(W) \cap (Y \cup V(T))$ can be extended to a $(V(W) \cap (Y \cup V(T)))$-good 3-coloring c_W of W. We combine c_T with the colorings c_W for all "non-root pieces" W to obtain a 3-coloring c of G. Note that every monochromatic component of c intersecting different pieces of the "decomposition" must meet $V(T) \cap V(W)$ for some "non-root piece" W. Then the superaddivitity of quadratic functions shows that c is Y-good. This finishes the outline of the proof of Theorem 2.1.

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