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Weak regularity and finitely forcible graph limits

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Abstract

Graphons are analytic objects representing limits of convergent sequences of graphs. Lovász and Szegedy conjectured that every finitely forcible graphon, i.e., a graphon determined by finitely many subgraph densities, is simple structured. In particular, one of their conjectures would imply that every finitely forcible graphon has a weak ε-regular partition with the number of parts bounded by a polynomial in ε^{-1}. We construct a finitely forcible graphon W such that the number of parts in any weak

ε-regular partition of W is at least exponential in $\varepsilon^{-2}/2^{5\log^* \varepsilon^{-2}}$. This bound almost matches the known upper bound and, in a certain sense, is the best possible.

Keywords: Graph limits, finitely forcible graphons, weak regularity

1 Introduction

Analytic methods applied to combinatorial limits has led to substantial results in many areas of mathematics and computer science, particularly in extremal combinatorics. A sequence of graphs $(G_n)_{n\in\mathbb{N}}$ is *convergent* if the sequence $d(H, G_n)$ converges for every graph H, where $d(H, G)$ be the *density* of a graph H in G, i.e., the probability that $|H|$ randomly chosen vertices of G induce a subgraph isomorphic to H, where $|H|$ is the order of H.

A *graphon* W is a symmetric measurable function from $[0,1]^2$ to $[0,1]$, i.e., $W(x,y) = W(y,x)$ for every $x, y \in [0,1]$. A *W-random graph* of order k is obtained by sampling k random points $x_1, \dots, x_k \in [0,1]$ and joining the i-th and the j-th vertex by an edge with probability $W(x_i, x_j)$. The *density* of a graph H in W is the probability that a W-random graph of order $|H|$ is H. If $(G_n)_{n\in\mathbb{N}}$ is a convergent sequence, then there exists a graphon W such that $d(H, W) = \lim_{n\to\infty} d(H, G_n)$ for every graph H. The graphon W can be viewed as the limit of $(G_n)_{n\in\mathbb{N}}$ and is unique in the sense given in [1]. Also see [6].

A graphon W is *finitely forcible* if there exist graphs $H_1, \dots, H_k$ such that if a graphon W' satisfies that $d(H_i, W') = d(H_i, W)$ for $i = 1, \dots, k$, then $d(H, W') = d(H, W)$ for every graph H. The following conjecture [7, Conjecture 7] links such graphons to extremal combinatorics.

Conjecture 1.1 *Let $H_1, \dots, H_k$ be finite graphs and $\alpha_1, \dots, \alpha_k$ reals. There exists a finitely forcible graphon W that minimizes the sum $\sum_{i=1}^{k} \alpha_i d(H_i, W)$.*

In [7], Lovász and Szegedy carried out a systematic study of finitely forcible graphons. The examples of finitely forcible graphons that they found led them

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to conjecture that all such graphons are simple structured [7, Conjectures 9 and 10], which was disproved through counterexample constructions in [4, 5].

Conjecture 10 from [7] is a starting point of our work. Analogously to weak regularity of graphs, every graphon has a weak ε-regular partition with at most $2^{O(\varepsilon^{-2})}$ parts. If Conjecture 10 from [7] were true, then every finitely forcible graphon would have weak ε-regular partitions with the number of parts polynomial in ε^{-1}. The number of parts in such partitions of the graphon constructed in [4] is $2^{\Theta(\log^2 \varepsilon^{-1})}$, much less than the general upper bound. We construct a finitely forcible graphon almost matching the upper bound.

Theorem 1.2 *There exist a finitely forcible graphon W and positive reals ε_i tending to 0 such that every weak ε_i-regular partition of W has at least $2^{\Omega\left(\varepsilon_i^{-2}/2^{5\log^* \varepsilon_i^{-2}}\right)}$ parts.*

As pointed out to us by Jacob Fox, there is no graphon (finitely forcible or not) matching the upper bound for infinitely many values of ε tending to 0.

Theorem 1.3 *There exist no graphon W, $c > 0$ and positive reals ε_i tending to 0 such that every weak ε_i-regular partition of W has at least $2^{c\varepsilon_i^{-2}}$ parts.*

In view of Theorem 1.3, Theorem 1.2 is almost the best possible.

2 Weak regular partitions of graphons

Several types of regular partitions of graphs derived from the original notion of Szemerédi exist. The notion of weak regular partitions from [3] is the most relevant for graphons. If W is a graphon, the *density* $d_W(A, B)$ between two measurable subsets A and B of $[0, 1]$ is the integral of W over $A \times B$. A partition of $[0, 1]$ into measurable sets $U_1, \ldots, U_k$ is weak ε-*regular* if

$$\left| d_W(A, B) - \sum_{i,j=1}^{k} \frac{d_W(U_i, U_j)|U_i \cap A||U_j \cap B|}{|U_i||U_j|} \right| \leq \varepsilon$$

for every two measurable subsets A and B where $|X|$ is the measure of X.

Naturally, one seeks weak regular partitions with few parts. The proof of the upper bound extends to graphons: for every $\varepsilon > 0$, there exists $k_0 \leq 2^{O(\varepsilon^{-2})}$ such that every graphon has an ε-regular partition with at most k_0 parts. Lovász and Szegedy [8] showed that the bound on k_0 must be at least $2^{\Omega(\varepsilon^{-1})}$, which was improved to $2^{\Omega(\varepsilon^{-2})}$ by Conlon and Fox [2], which matches the upper bound up to a multiplicative constant in the exponent.

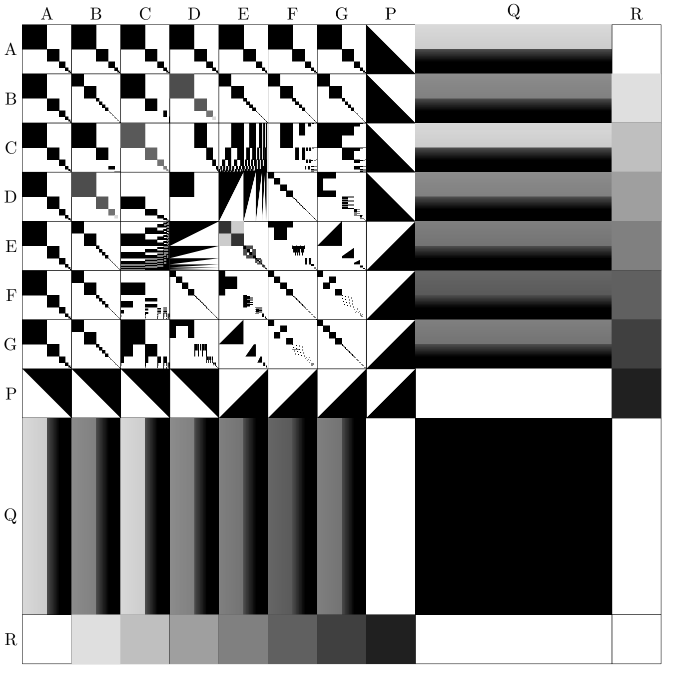

Fig. 1. The graphon constructed in Theorem 1.2.

The probabilistic construction of Conlon and Fox gives a step graphon W_ε such that every weak ε-regular partition of W_ε has at least $2^{\Omega(\varepsilon^{-2})}$ parts. However, an explicit construction of a different (step) graphon, which we denote by W_m^{CF}, can be distilled from [2]. In fact, a W_m^{CF}-random graph of order $2^{\alpha m}$ for α close to 0 gives the random graph construction from [2].

Since W_m^{CF} is a step graphon with 2^m parts, the number of parts of its weak ε-regular partition of W_m^{CF} never exceeds 2^m regardless the value of ε. A natural approach to obtain a (not necessarily finitely forcible) graphon not depending on ε with no weak ε-regular partition with fewer than $2^{\Theta(\varepsilon^{-2})}$ parts is to look at the limit of the sequence of graphons W_m^{CF}. This sequence is convergent but its limit is the graphon equal to 1/2 everywhere, which has weak regularity partitions with a single part. So, this approach fails.

3 Construction

The proof of finite forcibility of the graphon W from Theorem 1.2 uses the methods developed in [5] and further extended in [4]. The graphon is depicted in Figure 1; its several page long definition is omitted due to space limitations.

The most important part of the graphon is inside the part E. Let $t(n)$ be the tower function defined as $t(1) = 1$ and $t(n) = 2^{t(n-1)}$ if $n \geq 2$. For every $n \in \mathbb{N}$, the part E contains a copy of the graphon $W^{CF}_{t(n)}$ scaled to 2^{-n-1} of the size of this part. Since the copy of $W^{CF}_{t(n)}$ has no weak ε-regular partition with fewer than $2^{t(n)/4}$ parts if $\varepsilon < \frac{1}{2^{27+2n}t(n)^{1/2}}$, we get the bound in Theorem 1.2.

Theorem 1.3 implies that it is not possible to remove $2^{5\log^* \varepsilon_i^{-2}}$ completely from the denominator. However, our construction can be modified to replace $t(n)$ with faster growing functions of n, e.g., with $t(t(n))$, which would replace the function $2^{5\log^* \varepsilon_i^{-2}}$ with a slower growing function of ε^{-1}.

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