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Guarding Polyominoes, Polycubes and Polyhypercubes

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Abstract

We consider variations of the original art gallery problem where the domain is a polyomino, a polycube, or a polyhypercube. An m-polyomino is the connected union of m unit squares called pixels, an m-polycube is the connected union of m unit cubes called voxels, and an m-polyhypercube is the connected union of m unit hypercubes in a d dimensional Euclidean space. In this paper we generalize and unify the known results about guarding polyominoes and polycubes and obtain simpler proofs. We also obtain new art gallery theorems for guarding polyhypercubes.

Keywords: art gallery theorem, polyomino, polycube, polyhypercube.

1 Introduction

The original art gallery problem, posed by Klee in 1973, asks to find the minimum number of guards sufficient to cover any polygon with n vertices. The first solution to this problem was given by Chvátal [2], who proved that $\lfloor n/3 \rfloor$ guards are sometimes necessary, and always sufficient to cover a polygon with

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n vertices. Later Fisk [3] provided a shorter proof of Chvátal's theorem using an elegant graph coloring argument. Klee's art gallery problem has since grown into a significant area of study. Numerous *art gallery problems* have been proposed and studied with different restrictions placed on the shape of the galleries or the powers of the guards. (See the monograph by O'Rourke [8], and the surveys by Shermer [10] and Urrutia [11].)

In this paper we consider variations of the art gallery problem where the gallery is an *m-polyomino*, consisting of a connected union of m 1×1 unit squares called *pixels*, or an *m-polycube*, consisting of a connected union of m $1 \times 1 \times 1$ unit cubes called *voxels*. We will also consider higher dimensional cases where an m- polyhypercube is the connected union of m unit hypercubes in a d dimensional Euclidean space. Throughout this paper P_m denotes an m-polyomino when $d = 2$, an m-polycube when $d = 3$, or an m-polyhypercube when d is not specified. We say that a point $a \in P_m$ *covers* a point $b \in P_m$ provided $a = b$, or the line segment ab does not intersect the exterior of P_m. We say that a pixel/voxel A *covers* a point b, provided some point $a \in A$ covers b. A set of points $\mathcal{G}$ is called a *point guard set* for P_m if for every point $b \in P_m$ there is a point $a \in \mathcal{G}$ such that a covers b. A set of pixels/voxels $\mathcal{G}$ is called a *pixel/voxel guard set* for P_m if for every point $b \in P_m$ there is a pixel/voxel $A \in \mathcal{G}$ such that A covers b.

In [4], Irfan et al. show that $\lfloor \frac{m+1}{3} \rfloor$ point guards are always sufficient and sometimes necessary to cover any m-polyomino P_m, with $m \geq 2$. (See also Biedl et al. [1] for a detailed proof by case analysis.) Recently, Massberg [6] provided an alternate proof using perfect graphs. In [5], Iwerks claims that the same bound holds for polycubes and asks whether the result extends to polyhypercubes in $d \geq 4$ dimensions. In Section 2 we unify and generalize all these results proving that $\lfloor \frac{m+1}{3} \rfloor$ point guards are always sufficient and sometimes necessary to cover any m-polyhypercube P_m, with $m \geq 2$ in any dimension $d \geq 2$. While our lower bound example is a straight forward generalization of the examples in 2 and 3 dimensions, our argument for the upper bound is simpler than previous arguments, and works in every dimension.

In [9], Pinciu shows that $\lfloor \frac{m+1}{11} \rfloor + \lfloor \frac{m+5}{11} \rfloor + \lfloor \frac{m+9}{11} \rfloor$ pixel guards are always sufficient and sometimes necessary to cover an m-polyomino. Lower bounds and upper bounds for the number of voxels required to cover an m-polycube in 3D can be found in [5], but no sharp bounds are currently known when the dimension $d \geq 3$. In Section 3 we provide lower bounds for the number

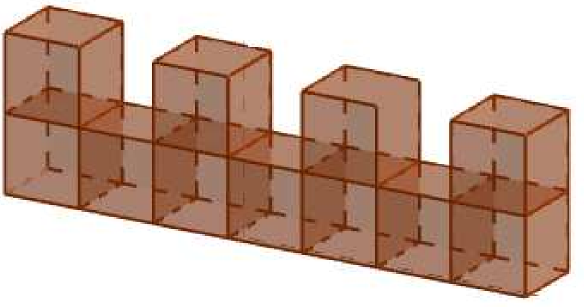

Fig. 1. An 11-polycube that requires 4 point guards.

of pixel/voxels required to cover an m-polyhypercube in d dimensions. Our bounds are dependent on d, and we conjecture that they are sharp.

In Section 4 we provide upper bounds independent of d for the number of pixel/voxel guards required to cover an m-polyhypercube.

2 Point Guards in PolyHypercubes

Here is our main result:

Theorem 2.1 *For any m-polyhypercube P_m with $m \geq 2$ in $d \geq 2$ dimensions, $\lfloor \frac{m+1}{3} \rfloor$ point guards are always sufficient, and sometimes necessary to cover P_m.*

Proof. We will use a construction to prove the necessity part of our result. First we will construct a polyhypercube P_m when $m \geq 2$ and $m \equiv 2 \bmod 3$. Then $m = 3k + 2$ for some non-negative integer k. For every integer i, $1 \leq i \leq 2k + 1$, we consider the hypercubes A_i that are bounded by the hyperplane $x_1 = i - 1$, the hyperplane $x_1 = i$, and the hyperplanes $x_j = 0$ and $x_j = 1$ where $j \neq 1$. For every odd integer i, $1 \leq i \leq 2k + 1$ we consider the hypercubes B_i on top of A_i. (obtained from a 1 unit translation of A_i along the x_d-axis.) An illustration of such a polyhypercube when $d = 3$ is shown in Figure 2. The constructed polyhypercube has $m = 3k + 2$ hypercubes and requires $\lfloor \frac{m+1}{3} \rfloor = k + 1$ points guards to be covered, as no two distinct hypercubes B_i's are visible by the same point guard. Simple alterations of this construction can provide examples in the case when $m \not\equiv 2 \bmod 3$. Sufficiency follows from Proposition 2.2 by selecting the guard set of smallest cardinality.□

Proposition 2.2 *For any m-polyhypercube P_m, there exist three point guard sets $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ such that:*

$$|\mathcal{G}_1| + |\mathcal{G}_2| + |\mathcal{G}_3| \leq m + 1.$$

Proof. Given an m-polyhypercube P_m in dimension $d \geq 2$, let G^*_m be the dual graph of P_m. Let T_m be a spanning tree of G^*_m. We can assume that T_m was constructed using the DFS or BFS algorithms, therefore T_m is the underlying graph of a rooted tree. Let A_1, A_2,...,A_m be the hypercubes of P_m in the order in which they are added while constructing T_m. We can assume without loss of generality that we chose a coordinate system such that all vertices have integer coordinates. Let V_1 be the set of vertices of P_m such that all d coordinates are odd, and let V_2 be the set of vertices of P_m such that all d coordinates are even. Then every hypercube of P_m has exactly one vertex in V_1 and one vertex in V_2. We will use Algorithm 1 to construct the three point guard sets $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$. Now it is easy to see the union between

Algorithm 1 Construction of $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$.

procedure
    $\mathcal{G}_1$:=the unique vertex of A_1 that is in V_1
    $\mathcal{G}_2$:=the unique vertex of A_1 that is in V_2
    $\mathcal{G}_3 := \emptyset$
    for $i := 2$ **to** m **do**
        let A_j with $j < i$ be the parent of A_i in the construction of T_m.
        let u be the unique vertex of A_j in V_1.
        let v be the unique vertex of A_j in V_2.
        let w be the unique vertex of A_i in $V_1 \cup V_2$ such that $w \neq u$ and $w \neq v$.
        choose distinct integers $k, l \in \{1, 2, 3\}$ such that $u \in \mathcal{G}_k$ and $v \in \mathcal{G}_l$. (such integers might not be unique.)
        $\mathcal{G}_{6-k-l} := \mathcal{G}_{6-k-l} \cup \{w\}$
    end for
end procedure

the hypercube A_i and its parent A_j is connected and covered by each of the sets $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$. Since A_i was arbitrary, we obtain that $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ are point guard sets of P_m. Moreover, since the cardinality of $|\mathcal{G}_1| + |\mathcal{G}_2| + |\mathcal{G}_3|$ was initially 2, and it can go up by at most 1 during each step of the loop, we obtain that $|\mathcal{G}_1| + |\mathcal{G}_2| + |\mathcal{G}_3| \leq m + 1$, which concludes the proof of the proposition. □

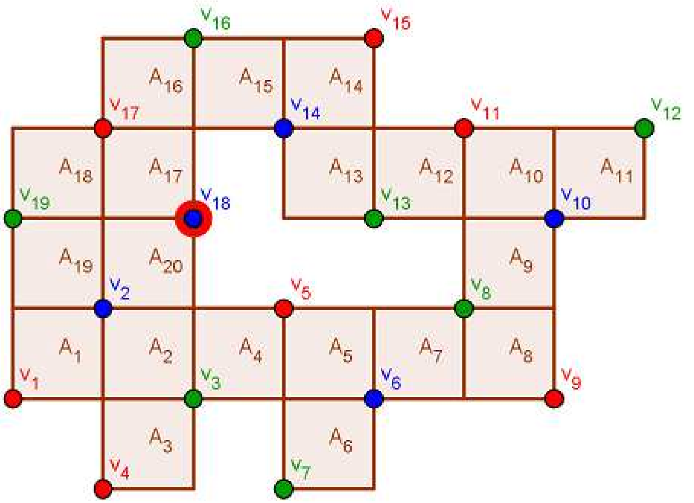

Fig. 2. Algorthm 1 applied to a 20-polyomino.

Finaly we would like to note that if G_m^* is a tree, then $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ are disjoint. However if G_m^* has cycles, then the same vertex can belong to more than one of the three guard sets. For example, if we apply the algorithm to the 20-polyomino from Figure 2 we obtain the following point guard sets: $\mathcal{G}_1 = \{v_1, v_4, v_5, v_9, v_{11}, v_{15}, v_{17}, v_{18}\}$, $\mathcal{G}_2 = \{v_2, v_6, v_{10}, v_{18}\}$ and $\mathcal{G}_3 = \{v_3, v_7, v_8, v_{12}, v_{13}, v_{16}, v_{19}\}$.

3 Voxel Guards in PolyHypercubes: Bounds Dependent of d

The following theorem provides a lower bound for the number of pixel/voxel guards required to cover all m-polyhypercubes in d dimensions:

Theorem 3.1 *For any integer $d \geq 2$ and for any integer $m \geq 2$ there exists an m-polyhypercube P_m in d dimensions such that the minimum number of pixel/voxel guards necessary to cover P_m is:*

$$\sum_{i=1}^{2d-3} \left\lfloor \frac{m+3i-2}{6d-1} \right\rfloor + \left\lfloor \frac{m+6d-7}{6d-1} \right\rfloor + \left\lfloor \frac{m+6d-3}{6d-1} \right\rfloor.$$

Proof. First we will construct P_m when $m > 6d$ and $m \equiv 2 \bmod(6d-1)$.

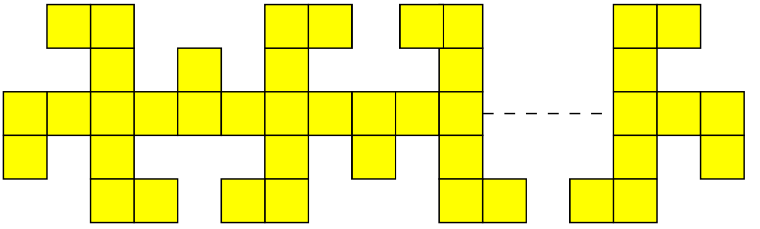

Fig. 3. An $(11k+2)$-polyomino that requires $3k+1$ pixel guards.

Then $m = (6d-1)k+2$ for some positive integer k. For every integer i, $1 \leq i \leq k$ we consider the hypercubes A_i that are bounded by the hyperplane $x_1 = 4i-4$, the hyperplane $x_1 = 4i-3$, and the hyperplanes $x_j = 0$ and $x_j = 1$ where $j \neq 1$. For every $1 \leq i \leq k-1$ we connect A_i and A_{i+1} with T-shaped 4-polyhypercubes. Then on every face of A_i that is not attached to a T-shaped polyhypercube we attach an L-shaped 3-polyhypercube to obtain P_m. Since there are $(k-1)$ T-shaped polyhypercubes and $(2d-2)k+2$ L-shaped polyhypercubes, then the number of hypercubes of P_m is:

$$m = k + 4(k-1) + 3[(2d-2)k+2] = (6d-1)k+2.$$

The polyomino P_{11k+2} from Figure 3 has $3+7k+4(k-1)+3 = 11k+2$ pixels ilustrates this construction when $d=2$. The dual graph of this polyhypercube is a tree with $(k-1)+[(2d-2)k+2] = (2d-1)k+1$ leaves. Since two pixels/voxels that correspond to a leaf cannot be guarded by the same pixel/voxel guard, then the number of pixels/voxels required to guard P_m is the same as the one stated in the theorem. Slight alterations of this construction obtained by deleting one to $6d-1$ hypercubes provide examples when $m \leq 6d$ or $m \not\equiv 2 \mod (6d-1)$. □

We conjecture that the bound from Theorem 3 is sharp:

Conjecture 3.2 *For any m-polyhypercube P_m in dimension $d \geq 2$ with $m \geq 2$,*

$$\sum_{i=1}^{2d-3} \left\lfloor \frac{m+3i-2}{6d-1} \right\rfloor + \left\lfloor \frac{m+6d-7}{6d-1} \right\rfloor + \left\lfloor \frac{m+6d-3}{6d-1} \right\rfloor$$

pixel/voxel guards are always sufficient, and sometimes necessary to cover P_m.

The conjecture is true when $d=2$, and the proof can be found in [9].

Theorem 3.3 *For any m-polyomino P_m with $m \geq 2$, $\lfloor \frac{m+1}{11} \rfloor + \lfloor \frac{m+5}{11} \rfloor + \lfloor \frac{m+9}{11} \rfloor$ pixel guards are always sufficient, and sometimes necessary to cover P_m.*

4 Voxel Guards in PolyHypercubes: Bounds Independent of d

One can notice that the sharp bound for the number of point guards sufficient to guard any m-polyhypercube is independent of the dimension d, while the bounds for the number of pixel/voxel guards are dependent of d. The following theorem gives us an upperbound for the number of pixel/voxel guards that depends on the number of hypercubes only, and is independent of d:

Theorem 4.1 (a) *For any m-polyhypercube P_m with $m \geq 3$ in $d \geq 2$ dimensions, $\lfloor \frac{1}{3}m \rfloor$ pixel/voxel guards are always sufficient to cover P_m.*

(b) *For any positive real number $c < \frac{1}{3}$, there exist positive integers m and d and an m-polyhypercube P_m which requires more than $\lfloor cm \rfloor$ pixel/voxel guards.*

Proof.

(a) It follows easily from the fact that the 2-domination number of the dual graph G_m^* of P_m is no more than $\lfloor \frac{1}{3}m \rfloor$ (see Meir et al. [7]), and the fact that two hypercubes that correspond to vetices within distance two in G_m^* cover each other. An alternate proof can be obtained by applying Theorem 2.1 to the polyhypercube obtained from P_m by deleting a hypercube that corresponds to a leaf in a spanning tree of G_m^*.

(b) The m-polyhypercubes constructed in Theorem 3.1 require

$$\sum_{i=1}^{2d-3} \left\lfloor \frac{m+3i-2}{6d-1} \right\rfloor + \left\lfloor \frac{m+6d-7}{6d-1} \right\rfloor + \left\lfloor \frac{m+6d-3}{6d-1} \right\rfloor \approx \left\lfloor \frac{2d-1}{6d-1} m \right\rfloor$$

pixel/voxel guards to be covered. If $c < \frac{1}{3}$ and d is sufficiently large, then $\frac{2d-1}{6d-1} > c$, therefore P_m requires more than $\lfloor cm \rfloor$ pixel/voxel guards.

□

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