Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in DISCRETE MATHEMATICS

www.elsevier.com/locate/endm

Even pairs in square-free Berge graphs

Frédéric Maffray [1]

CNRS, Laboratoire G-SCOP, Université de Grenoble-Alpes
Grenoble, France

Abstract

This is a short example to show the basics of using the ENDM style macro files. Ample examples of how files should look may be found among the published volumes of the series at the ENDM home page (http://www.elsevier.com/locate/endm)

Keywords: Please list keywords for your paper here, separated by commas.

1 Introduction

A graph G is *perfect* if every induced subgraph H of G satisfies $\chi(H) = \omega(H)$, where $\chi(H)$ is the chromatic number of H and $\omega(H)$ is the maximum clique size in H. A *hole* is a chordless cycle with at least four vertices and an *antihole* is the complement of a hole. A *Berge graph* is any graph that contains no odd hole and no odd antihole of length at least 5. Berge's famous "Strong Perfect Graph Conjecture" [1,2,3,12] was solved by Chudnovsky, Robertson, Seymour and Thomas [6]: *Every Berge graph is perfect.* Moreover, Chudnovsky, Cornuéjols, Liu, Seymour and Vušković [5] devised a polynomial-time algorithm that determines if a graph is Berge. It is known that one can obtain an optimal coloring of a perfect graph in polynomial time due to the algorithm

[1] E-mail: frederic.maffray@grenoble-inp.fr. Partially supported by ANR project STINT under reference ANR-13-BS02-0007.

of Grötschel, Lovász and Schrijver [9]. This algorithm however is not purely combinatorial and is generally considered impractical.

An *even pair* in a graph G is a pair $\{x, y\}$ of non-adjacent vertices such that every chordless path between them has even length (number of edges). A graph G is *even-contractile* [4] if either G is a clique or there exists a sequence $G_0, \dots, G_k$ of graphs such that $G = G_0$, for $i = 0, \dots, k-1$ the graph G_i has an even pair $\{x_i, y_i\}$, G_{i+1} is the graph obtained from G_i by contracting (identifying) x_i and y_i into one vertex, and G_k is a clique. A graph G is *perfectly contractile* if every induced subgraph of G is even-contractile. This class is of interest because many classical families of graphs are perfectly contractile and consequently admit a "purely combinatorial" coloring algorithm; see [8].

Everett and Reed [8] proposed a conjecture aiming at a characterization of perfectly contractile graphs. A *prism* is a graph that consists of two vertex-disjoint triangles (cliques of size 3) with three vertex-disjoint paths P_1, P_2, P_3 between them, and with no other edge than those in the two triangles and in the three paths. The length of a path is its number of edges. Note that if two of the paths P_1, P_2, P_3 have lengths of different parities, then their union induces an odd hole. So in a Berge graph, the three paths of a prism have the same parity. A prism is *even* (resp. *odd*) if these three paths all have even lengths (resp. all have odd lengths).

Conjecture 1.1 **([8])** *A graph is perfectly contractile if and only if it contains no odd hole, no antihole of length at least* 5*, and no odd prism.*

Conjecture 1.2 **([8])** *Any graph that contains no odd hole, no antihole of length at least* 5*, and no prism is perfectly contractile.*

Conjecture 1.1 remains open. Let $\mathcal{A}$ be the class of graphs that contain no odd hole, no antihole of length at least 5, and no prism. Conjecture 1.2 was proved in [11], as follows.

Theorem 1.3 **([11])** *There is a polynomial time algorithm which returns a clique of size $\omega(G)$ and a coloring with $\omega(G)$ colors for every graph G in $\mathcal{A}$.*

A *square* is a hole of length four. A graph is *square-free* if it does not contain a square as an induced subgraph. We will be able to prove that every square-free Berge graph with no odd prism either is a clique or has an even pair, as suggested in Conjecture 1.1. Unfortunately, contracting an even pair may result in the presence of a square in the contracted graph, so the result does not yield a proof of Conjecture 1.1 for square-free graphs. Nevertheless, using the presence of even pairs, we can prove the following theorem.

Theorem 1.4 *There exists a combinatorial and polynomial time algorithm which, given any square-free Berge graph G with no odd prism, returns an $\omega(G)$ coloring of G and a clique of size $\omega(G)$.*

Since Theorem 1.3 settles the case of graphs that have no prism, we may assume from now on that we are dealing with a square-free Berge graph G that contains no odd prism and contains an even prism.

2 Hyperprisms

Given a set $T \subset V(G)$, we say that a vertex of $V(G) \setminus T$ is *complete* to T if it is adjacent to all vertices of T. Given two sets $S, T \subset V(G)$, S is *complete* to T if every vertex of S is complete to T.

As in [6], a *hyperprism* is a 9-tuple $(A_1, C_1, B_1, A_2, C_2, B_2, A_3, C_3, B_3)$ of non-empty and pairwise disjoint subsets of $V(G)$ with the following properties:

- For distinct $i, j \in \{1, 2, 3\}$, A_i is complete to A_j, and B_i is complete to B_j, and there are no other edges between $A_i \cup C_i \cup B_i$ and $A_j \cup C_j \cup B_j$.
- For each $i \in \{1, 2, 3\}$, every vertex of $A_i \cup C_i \cup B_i$ belongs to a chordless path between A_i and B_i with interior in C_i.

For each $i \in \{1, 2, 3\}$, a chordless path from A_i to B_i with interior in C_i is called an *i-rung*. Let us write $A = A_1 \cup A_2 \cup A_3$, $B = B_1 \cup B_2 \cup B_3$ and $C = C_1 \cup C_2 \cup C_3$. Let $S_i = A_i \cup B_i \cup C_i$ for $i \in \{1, 2, 3\}$. The triple (A_i, C_i, B_i) is called a *strip* of the hyperprism. We call (A, C, B) the *profile* of the hyperprism. Given two hyperprisms η and η' with profiles (A, C, B) and (A', C', B') respectively, we write $\eta \prec \eta'$ if $C \subseteq C'$ and either (i) $A \subseteq A'$ and $B \subseteq B'$ or (ii) $A \subseteq B'$ and $B \subseteq A'$ and one of these inclusions is strict. Clearly, $\prec$ is an order relation on hyperprisms, so we can speak about maximal hyperprisms for $\prec$.

Let η be a maximal hyperprism. Let H be the subgraph of G induced on the union of the nine sets A_i, B_i, C_i, $i = 1, 2, 3$. If we pick any i-rung R_i for each $i \in \{1, 2, 3\}$, with ends $a_i \in A_i$ and $b_i \in B_i$, we see that R_1, R_2, R_3 form a prism K; any such prism is called an *instance* of the hyperprism. A vertex x in $V(G) \setminus V(K)$ is a *major* neighbor of K if x has at least two neighbors in $\{a_1, a_2, a_3\}$ and at least two neighbors in $\{b_1, b_2, b_3\}$. A vertex x in $V(G) \setminus V(H)$ is a *major* neighbor of H if x is a major neighbor of an instance of η. Let M be the set of all major neighbors of H.

Lemma 2.1 *The following properties hold:*
- *M is a clique.*

- *For each* $i \in \{1,2,3\}$, $M \cup A_i \cup B_i$ *is a cutset that separates* C_i *from* $S_{i+1} \cup S_{i+2}$.
- *Two of* A_1, A_2, A_3 *and two of* B_1, B_2, B_3 *are cliques.*
- M *is complete to at least two of* A_1, A_2, A_3 *and at least two of* B_1, B_2, B_3.
- *There is an integer* $j \in \{1,2,3\}$ *such that* A_j *and* B_j *are cliques and* M *is complete to* $A_j \cup B_j$.

2.1 Selecting a strip

Let us say that a strip (A_i, C_i, B_i) of the hyperprism is *good* if both A_i and B_i are cliques and M is complete to $A_i \cup B_i$. Lemma 2.1 says that every maximal hyperprism in G has a good strip. We may assume that (A_1, C_1, B_1) is a good strip of η. Moreover, we choose η such that S_1 has the smallest size over all good strips of maximal hyperprisms.

Let R' and R'' be two 1-rungs of η, where R' has ends u', w, and R'' has ends u'', w, and $u' \neq u''$ (so w is in one of the two sets A_1, B_1 and u', u'' are in the other set). We say that R' and R'' *converge* if u' has no neighbor in $R'' \setminus u''$ and u'' has no neighbor in $R' \setminus u'$.

Lemma 2.2 *There do not exist two* 1*-rungs that converge.*

Proof. In the opposite case, we are able to construct a maximal hyperprism η' that has a smaller good strip than η, a contradiction to the choice of η. □

2.2 Finding an even pair

Consider any $b \in B_1$. For any two $a, a' \in A_1$, write $a <_b a'$ whenever there exists an odd chordless path R from a to b such that a' is the neighbor of a on R. For each $a \in A_1$ define similarly a relation $<_a$ on B_1.

Lemma 2.3 *For each* $b \in B_1$, $<_b$ *is an order relation.*

Proof. In the opposite case, we are able to find two 1-rungs that converge. □

Using Lemma 2.3 and its analogue for every vertex in $A_1 \cup B_1$, we can establish the following.

Lemma 2.4 *There exists an even pair* $\{a, b\}$ *of* G *with* $a \in A_1$ *and* $b \in B_1$.

Let $A_1 = \{a_1, \dots, a_k\}$ and $B_1 = \{b_1, \dots, b_\ell\}$, and assume that $k \leq \ell$. By Lemma 2.4, we may assume that $\{a_1, b_1\}$ is an even pair of G; and similarly, for $i = 2, \dots, k$, we may assume that $\{a_i, b_i\}$ is an even pair of $G \setminus \{a_1, b_1, \dots, a_{i-1}, b_{i-1}\}$.

2.3 Decomposing the graph

By Lemma 2.1, $V(G) \setminus (M \cup A_1 \cup B_1)$ can be partitioned into two subsets X and Y, with $C_1 \subseteq X$ and $C_2 \subset Y$, such that there is no edge between X and Y. Let $G_X = G \setminus Y$ and $G_Y = G \setminus X$. Thus we consider that G is decomposed into G_X and G_Y. Since G_X and G_Y are proper induced sugraphs of G, we may assume by induction that we have a clique Q_X of G_X of size $\omega(G_X)$ and a coloring c_X of G_X with $\omega(G_X)$ colors, and the same for G_Y.

Lemma 2.5 *There exists a coloring c'_X of G_X with $\omega(G_X)$ colors such that $c'_X(a_i) = c'_X(b_i)$ for all $i = 1, \dots, k$, and such a coloring can be obtained from c_X in polynomial time.*

Proof. Suppose that c_X itself does not have the desired property, and let h be the smallest integer such that $c_X(a_h) \neq c_X(b_h)$. In case $h > 1$, we may assume, up to relabeling, that $c_X(a_i) = i = c_X(b_i)$ for all $i = 1, \dots, h-1$. Let $c_X(a_h) = i$ and $c_X(b_h) = j$, with $i \neq j$. Let W be the bipartite subgraph of G induced by the vertices of color i and j. We swap colors i and j in the component of W that contains a_h. This component does not contain b_h, for otherwise it contains a chordless odd path between a_h and b_h, and this path is in $G \setminus \{a_1, b_1, \dots, a_{h-1}, b_{h-1}\}$ since it contains no vertex of color less than i and j; but this contradicts the fact that $\{a_h, b_h\}$ is an even pair of $G \setminus \{a_1, b_1, \dots, a_{h-1}, b_{h-1}\}$. So, after the swap, a_h and b_h have the same color. Thus we obtain a coloring of G_X with $\omega(G_X)$ colors where the value of h has increased. Repeating this procedure at most k times leads to the desired coloring. □

Applying Lemma 2.5 to both G_X and G_Y, we obtain colorings c_X and c_Y of G_X and G_Y respectively such that, up to relabeling, $c_X(a_i) = c_X(b_i) = c_Y(a_i) = c_Y(b_i)$ for each $i = 1, \dots, k$. By Lemma 2.1, $M \cup (B \setminus \{b_1, \dots, b_k\})$ is a clique and all its vertices are adjacent to b_i for each $i = 1, \dots, k$. So we may assume that every vertex z in $M \cup (B \setminus \{b_1, \dots, b_k\})$ satisfies $c_X(z) = c_Y(z)$. Thus the two colorings c_X and c_Y coincide on the cutset $M \cup A_1 \cup B_1$ and so they can be merged into a coloring of G. This coloring uses $\max\{\omega(G_X), \omega(G_Y)\}$ colors, and one of Q_X and Q_Y is a clique of that size. So the coloring and the larger of these two cliques are both optimal.

3 The algorithm

We can now describe our algorithm. Let $\mathcal{G}$ be the class of square-free Berge graphs with no odd prism.

Input: A graph G on n vertices.
Output: Either a coloring of G and a clique of the same size, or the negative answer "G is not in $\mathcal{G}$".
Step 1. Test whether G is square-free, and test whether G is Berge with the algorithm from [5]. If any of these tests fails, return the answer "G is not in $\mathcal{G}$" and stop. Now test whether G contains a prism as explained in [10]. If this algorithm produces an odd prism, then return the negative answer and stop; if it shows that G contains no prism, then color G applying the algorithm from [11].
Step 2. Now suppose that Step 1 has produced an even prism. Grow a maximal hyperprism η, and find a good strip S_1 of η.
Apply Lemma 2.3 to each vertex $x \in A_1 \cup B_1$. Its proof either establishes that $<_x$ is an order relation or finds 1-rungs that converge; in the latter case, the proof of Lemma 2.2 produces a new maximal hyperprism η' with a smaller good strip; then restart from η'.
When $<_x$ is an order relation for all $x \in A_1 \cup B_1$, Lemma 2.4 shows how to find even pairs. The graph G is decomposed into graphs G_X and G_Y, and an optimal coloring and a maximal clique for G can be obtained as explained above.

3.1 Complexity analysis

We can test whether a graph G is Berge in time $O(n^9)$ with the algorithm from [5]. We can test whether G is square-free in time $O(n^4)$, and whether a Berge graph G contains a prism in time $O(n^5)$ as explained in [10]. Now assume that the algorithm produces an even prism. It is easy to to see that all the procedures in Step 2 of the algorithm (growing a maximal hyperprism, determining the orderings) can be performed in time at most $O(n^3)$. We complete our analysis with two remarks.
(a) When we restart from a new hyperprism, the size of the good strip is strictly smaller; so this restarting step occurs at most $O(n)$ times.
(b) When G is decomposed into graphs G_X and G_Y, the algorithm is called recursively on them. This corresponds to a decomposition tree T for G: every non-leaf node of T is an induced subgraph G' of G and has two children which are induced subgraphs of G'; and every leaf of T is a graph that contains no prism. Let us show that this tree has polynomial size. When G is decomposed into graphs G_X and G_Y as above, because of a certain cutset that arises from a hyperprism η, we mark the corresponding node of the tree with a pair of vertices $\{c_1, c_2\}$ where $c_1 \in C_1$ and $c_2 \in C_2$ are chosen arbitrarily. We mark

every subsequent decomposition node similarly. We can show the following.

Lemma 3.1 *Every pair of vertices of G is used to mark at most one node of the decomposition tree.*

By Lemma 3.1 the total number of nodes in T is $O(n^2)$. The leaves of the decomposition tree T are Berge graphs with no antihole (since they are square-free) and no prism, so they can be colored in time $O(n^6)$ as explained in [11]. At each node G' of T different from the root G, we know that G' is an induced subgraph of G, so it is a square-free Berge graph; hence we must only test whether G' contains a prism, which is done in time $O(n^5)$ as explained in [10]. So the total complexity of the algorithm is $O(n^9 + n^2 \times n^5 + n^2 \times n^6) = O(n^9)$.

References

[1] C. Berge. *Les problèmes de coloration en théorie des graphes.* Publ. Inst. Stat. Univ. Paris 9 (1960) 123–160.

[2] C. Berge. *Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind (Zusammenfassung).* Wiss. Z. Martin Luther Univ. Math.-Natur. Reihe (Halle-Wittenberg) 10 (1961) 114–115.

[3] C. Berge. “Graphs.” North-Holland, Amsterdam/New York, 1985.

[4] M.E. Bertschi. *Perfectly contractile graphs.* J. Comb. Th. B 50 (1990) 222–230.

[5] M. Chudnovsky, G. Cornuéjols, X. Liu, P. Seymour, K. Vušković. *Recognizing Berge graphs.* Combinatorica 25 (2005) 143–186.

[6] M. Chudnovsky, N. Robertson, P. Seymour, R. Thomas. *The strong perfect graph theorem.* Annals of Mathematics 164 (2006) 51–229.

[7] M. Conforti, G. Cornuéjols, K. Vušković. *Square-free perfect graphs.* Journal of Combinatorial Theory B 90 (2004) 257–307.

[8] H. Everett, C.M.H. de Figueiredo, C. Linhares Sales, F. Maffray, O. Porto, B.A. Reed. *Even pairs.* In [12], 67–92.

[9] M. Grötschel, L. Lovász, A. Schrijver. *The ellipsoid method and its consequences in combinatorial optimization.* Combinatorica 1 (1981) 169–197.

[10] F. Maffray, N. Trotignon. *Algorithms for perfectly contractile graphs.* SIAM Journal on Discrete Mathematics 19 (2005) 553–574.

[11] F. Maffray, N. Trotignon. *A class of perfectly contractile graphs.* Journal of Combinatorial Theory B 96 (2006) 1–19.

[12] J.L. Ramírez-Alfonsín, B.A. Reed. “Perfect Graphs.” Wiley Interscience, 2001.