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Cycles and matchings in randomly perturbed digraphs and hypergraphs

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Abstract

We consider several situations where "typical" structures have certain spanning substructures (in particular, Hamilton cycles), but where worst-case extremal examples do not. In these situations we show that the extremal examples are "fragile" in that after a modest random perturbation our desired substructures will typically appear. This builds on a sizeable existing body of research. Our first theorem is that adding linearly many random edges to a dense k-uniform hypergraph typically ensures the existence of a perfect matching or a loose Hamilton cycle. We outline the proof of this theorem, which involves a nonstandard application of Szemerédi's regularity lemma to “beat the union bound”; this might be of independent interest. Our next theorem is that digraphs with certain strong expansion properties are pancyclic. This implies that adding a linear number of random edges typically makes a dense digraph pancyclic. Our final theorem is that perturbing a certain (minimum-degree-dependent) number of random edges in a tournament typically ensures the existence of multiple edge-disjoint Hamilton cycles. All our results are tight.

Keywords: hypergraph, digraph, tournament, smoothed analysis, perfect matching, Hamilton cycle, pancyclic

1 Introduction

We say that a graph is *Hamiltonian* if it has a *Hamilton cycle*: a simple cycle containing every vertex in the graph. Hamiltonicity is a central notion in graph theory and has been extensively studied in a wide range of contexts. In particular, due to a seminal paper by Karp [7], it has become a canonical NP-complete problem to determine whether an arbitrary graph is Hamiltonian. There are nevertheless a variety of easily-checkable conditions that guarantee Hamiltonicity. The most famous of these is given by a classical theorem of Dirac [3], which states that any n-vertex graph ($n \geq 3$) with minimum degree at least $n/2$ is Hamiltonian.

Dirac's theorem demands a very strong density condition, but in a certain asymptotic sense “almost all” dense graphs are Hamiltonian. If we fix $\alpha > 0$ and select a graph uniformly at random among the (labelled) graphs with n vertices and $\alpha\binom{n}{2}$ edges, then the degrees will probably each be about αn. Such a random graph is Hamiltonian with probability $1 - o(1)$ (we say it is Hamiltonian *asymptotically almost surely*, or *a.a.s.*). This follows from a stronger result [11,9] that gives a *threshold* for Hamiltonicity: a random n-vertex, m-edge graph is Hamiltonian a.a.s. if $m \gg n \log n$, and fails to be Hamiltonian a.a.s. if $m \ll n \log n$. Here and from now on, all asymptotics are as $n \to \infty$, and we implicitly round large quantities to integers.

In [2], the authors studied Hamiltonicity in the random graph model that starts with a fixed graph and adds m random edges (this model has since been studied in a number of other contexts; see for example [1,10]). They found that to ensure Hamiltonicity in this model we only need m to be linear, saving a logarithmic factor over the standard model where we start with nothing. To be precise, [2, Theorem 1] says that for every $\alpha > 0$ there is $c = c(\alpha)$ such that if we start with a graph with minimum degree at least αn and add cn random edges, then the resulting graph will a.a.s. be Hamiltonian. Note that some dense graphs require a linear number of extra edges to become Hamiltonian (consider the complete bipartite graph with partition sizes $n/3$ and $2n/3$), so the order of magnitude of this result is tight.

We can interpret this theorem as quantifying the "fragility" of the few dense graphs that are not Hamiltonian, by determining the amount of random perturbation that is necessary to make them Hamiltonian. This is comparable to the notion of *smoothed analysis* of algorithms introduced in [13], which involves studying the performance of algorithms on randomly perturbed inputs. The pioneering result in this field explained why the simplex algorithm is efficient in practice: even though the algorithm may perform poorly on certain pathological inputs, these worst-case inputs are not robust under the small amount of random noise likely to exist in the real world. Similarly, the previous theorem suggests that if a dense graph is not too rigidly "structured", then we can expect it to be Hamiltonian, even if there is no reason for it to be "typical" among all dense graphs. We note that in this context it may be more natural to consider different models of random perturbation, in particular those that delete as well as add random edges. But in most cases, it is very easy to transfer theorems between different models: since deletion of few edges will not destroy the density of a graph, it is the addition of edges that is really important.

Our first contribution is to generalize the aforementioned theorem to hypergraphs (and to give a corresponding result for perfect matchings, which is nontrivial in the hypergraph setting). Unfortunately, there is no single most natural notion of a cycle or of minimum degree in hypergraphs. A k-uniform *loose* cycle is a k-uniform hypergraph with a cyclic ordering on its vertices such that every edge consists of k consecutive vertices and every pair of consecutive edges intersects in exactly one vertex. The degree of a set of vertices is the number of edges that include that set, and the *minimum* $(k-1)$*-degree* δ_{k-1} is the minimum degree among sets of $k-1$ vertices. Let $\mathbb{H}_k(n,m)$ be the uniform distribution on m-edge k-uniform hypergraphs on the vertex set $[n]$.

Theorem 1.1 *For each $\alpha>0$ there is $c=c(\alpha)$ such that:*

(a) If H is a k-uniform hypergraph on $[kn]$ with $\delta_{k-1}(H)\ge \alpha n$, and $R\in \mathbb{H}_k(kn,cn)$, then $H\cup R$ a.a.s. has a perfect matching.

(b) If H is a k-uniform hypergraph on $[(k-1)n]$ with $\delta_{k-1}(H)\ge\alpha n$, and $R\in\mathbb{H}((k-1)n,cn)$, then $H\cup R$ a.a.s. has a loose Hamilton cycle.

All the motivation for graphs is still relevant in the hypergraph setting. Dirac's theorem approximately generalizes to hypergraphs (see [8]): for small ε and large n, if the minimum $(k-1)$-degree of an n-vertex k-uniform hypergraph is greater than $(1/(2(k-1))+\varepsilon)\,n$ then that hypergraph contains a loose Hamilton cycle. Just as for graphs, the threshold for both perfect matchings and loose Hamilton cycles in k-uniform hypergraphs is $n\log n$ random edges (see [4] and [6, Corollary 2.6]), so "almost all" dense hypergraphs have Hamilton cycles and perfect matchings.

In Section 2 we will outline the ideas in the proof of Theorem 1.1. The methods usually employed to study Hamilton cycles and perfect matchings in random graphs are largely ineffective in the hypergraph setting, so we need a very different proof. In particular, we cannot easily manipulate paths for Pósa-type arguments. Our proof involves reducing the theorem to the a.a.s. existence of a perfect matching in the union of a dense bipartite graph G with a random almost-perfect matching M. The obvious naïve approach to prove this lemma would be to show that each vertex set expands in $G\cup M$ with high probability, and then apply the union bound over all such sets and finish the proof with Hall's theorem. However, the probabilities of failure to expand are not small enough for the union bound to work over all the exponentially many vertex subsets. In fact the "reason" for a perfect matching in this perturbed graph seems to be quite different depending on the structure of the initial bipartite graph. We therefore apply Szemerédi's regularity lemma to break up the graph into clusters of vertices each of which behave "roughly the same", and use the union bound to show only that the edges of M spread out well between the clusters. We can then combine the expansion properties of G within clusters, and the expansion properties of M between clusters, to prove the theorem.

We also present some other related theorems, without proof. Our second theorem gives a general expansion condition for *pancyclicity*. We say an n-vertex (di-)graph is pancyclic if it contains cycles of all lengths ranging from 3 to n.

Theorem 1.2 *Let D be a directed graph on n vertices with all in- and out-degrees at least $8k$, and suppose for every pair of disjoint sets $A, B\subseteq V(D)$ with $|A|=|B|\geq k$, there is an arc from A to B. Then D is pancyclic.*

We hope this theorem could be of independent interest, but our particular motivation is that it implies a number of results about randomly perturbed graphs and digraphs. In particular it provides very simple proofs of the theorems in [2] concerning Hamiltonicity in randomly perturbed graphs and digraphs, and allows us to extend these theorems to pancyclicity. Most generally, Theorem 1.2 implies the following theorem. Let $\mathbb{D}(n,m)$ be the uniform distribution on m-arc digraphs on the vertex set $[n]$.

Theorem 1.3 *For each $\alpha > 0$, there is $c = c(\alpha)$ such that if D is a digraph on $[n]$ with all in- and out- degrees at least αn, and $R \in \mathbb{D}(n, cn)$, then $D \cup R$ is a.a.s. pancyclic.*

Our final theorem concerns randomly perturbed tournaments. Although it is easy to construct tournaments with no Hamilton cycle, we have in fact shown that every tournament becomes Hamiltonian after a small random perturbation. We also show that randomly perturbed tournaments are not just Hamiltonian, but have multiple edge-disjoint Hamilton cycles. Moreover, we can give stronger results for tournaments with large minimum in- and out-degrees.

Theorem 1.4 *Consider a tournament T with n vertices and all in- and out-degrees at least d. Independently choose $m = \omega(n/(d+1))$ random edges of T and orient them uniformly at random. The resulting perturbed tournament P a.a.s. has q arc-disjoint Hamilton cycles, for $q = O(1)$.*

Note that we allow the minimum degree d to be an arbitrary (possibly zero) function of n.

2 Proof outline of Theorem 1.1

The ideas in the proofs of parts (a) and (b) of Theorem 1.1 are almost exactly the same; we outline only the proof of part (a) because the explanation is simpler.

First, we note that R typically has an almost-perfect matching on its own. To be precise, for any $\varepsilon > 0$, if c is large then R a.a.s. has a matching M of $(1-\varepsilon)n$ edges. (Note that a.a.s. each set of εkn vertices contains an edge, so such a matching can be chosen greedily).

By the symmetry of the distributon of R, we can assume that M is a uniformly random matching of its size. Randomly extend M to a perfect matching $\overline{M}$ on the vertex set $V(H)$, and randomly split each edge e_i of $\overline{M}$ into a vertex a_i and a $(k-1)$-set b_i. Let A be the set of all such a_i and let B contain all the b_i. Now, for any hypergraph L on $V(G)$, we define a bipartite graph $G_{A,B}(L)$ on the vertex set $A \cup B$, by putting an edge between a_i and b_j if $\{a_i\} \cup b_j$ is an edge of L. This means $G_{A,B}\left(\overline{M}\right)$ is a perfect matching and $G_{A,B}(M)$ is an almost-perfect sub-matching of $G\left(\overline{M}\right)$.

In fact, if we condition on A and B then $G_{A,B}(M)$ is a uniformly random almost-perfect matching between A and B. Since A and B are random, concentration inequalities show that $\delta(G_{A,B}(H)) = \Omega(n)$. Existence of a perfect matching in $H \cup R$ then reduces to the following lemma:

Lemma 2.1 *There is $\xi = \xi(\alpha) > 0$ such that the following holds. Let G be a bipartite graph with parts A, B of equal size n, and suppose $\delta(G) \geq \alpha n$. Let M be a uniformly random matching between A and B with $(1-\xi)n$ edges. Then a.a.s. $G \cup M$ has a perfect matching.*

Proof (Sketch) In order to apply Hall's theorem, we need to show that a.a.s. $|N_{G\cup M}(W)| \geq |W|$ for all $W \subseteq A$. The cases $|W| \leq \alpha n$ and $|W| \geq (1-\alpha)n$ are trivial (the density condition on G alone is enough). So we focus on the case where $\alpha n \leq |W| \leq (1-\alpha)n$.

For each such W (say $|W| = wn$ and $|N_G(W)| = \nu n$), if ξ is small then

$$\begin{aligned}\mathbb{E}\,|N_{M\cup G}(W)| &= \mathbb{E}\,|N_M(W)\setminus N_G(W)| + |N_G(W)| \\ &= w(1-\nu)(1-\xi)n + \nu n \\ &= wn + (\nu(1-w) - O(\xi))n \\ &\geq |W|.\end{aligned}$$

We can use concentration inequalities to show that $\Pr(|N_{M\cup G}(W)| < |W|) = e^{-\Theta(n)}$, but we cannot show that this probability is anywhere near as small as 2^{-n}. Therefore we cannot naïvely use the union bound over all subsets W.

So, we apply a certain form of Szemerédi's regularity lemma. We can partition almost all the vertices of A (respectively B) into $O(1)$ clusters, such that the edges between each pair of clusters are "random-like". This means that a not-too-small subset of a cluster has roughly the same adjacencies as the whole cluster.

The critical observation is that it now (more or less) suffices to consider subsets $W \subseteq A$ which are the union of complete clusters, and there are only $O(1)$ such W. This is because even if $W' \subseteq W$ has only partial intersection with the clusters, it still has almost the same adjacencies in G as the whole of W. That is to say, the edges of M that avoid neighbours of W' in G are almost the same as the edges that avoid neighbours of W. So (roughly speaking) the concentration of $|N_M(W)\setminus N_G(W)|$ is enough to give a bound on $|N_M(W')\setminus N_G(W')|$ that allows us to show $|N_{M\cup G}(W')| \geq |W'|$. □

3 Concluding remarks

There are a few natural questions that remain open. In particular, a k-uniform *tight* cycle is a k-uniform hypergraph with a cyclic ordering on its vertices such that every k consecutive vertices form an edge. There is a Dirac-type theorem for tight Hamilton cycles [12], and a random dense k-uniform hypergraph typically has a tight Hamilton cycle [5], so we might expect an analog of Theorem 1.1 to hold for tight Hamilton cycles. However, it appears this would require quite a different proof.

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