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Induced minors and well-quasi-ordering

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Abstract

A graph H is an induced minor of a graph G if it can be obtained from an induced subgraph of G by contracting edges. Otherwise, G is said to be H-induced minor-free. Robin Thomas showed in [*Graphs without* K_4 *and well-quasi-ordering*, Journal of Combinatorial Theory, Series B, 38(3):240 – 247, 1985] that K_4-induced minor-free graphs are well-quasi ordered by induced minors.

We provide a dichotomy theorem for H-induced minor-free graphs and show that the class of H-induced minor-free graphs is well-quasi-ordered by the induced minor relation if and only if H is an induced minor of the gem (the path on 4 vertices plus a dominating vertex) or of the graph obtained by adding a vertex of degree 2 to the complete graph on 4 vertices.

Similar dichotomy results were previously given by Guoli Ding in [*Subgraphs and well-quasi-ordering*, Journal of Graph Theory, 16(5):489–502, 1992] for subgraphs and Peter Damaschke in [*Induced subgraphs and well-quasi-ordering*, Journal of Graph Theory, 14(4):427–435, 1990] for induced subgraphs.

Keywords: Well-quasi-ordering, induced minors, combinatorial dichotomies.

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1 Introduction

A *well-quasi-order* (*wqo* for short) is a quasi-order which contains no infinite decreasing sequence, nor an infinite collection of pairwise incomparable elements (called an *antichain*). One of the most important results in this field is arguably the theorem by Robertson and Seymour which states that graphs are well-quasi-ordered by the minor relation [14]. Other natural containment relations are not so generous; they usually do not wqo all graphs. In the last decades, much attention has been brought to the following question: given a partial order $(S, \preceq)$, what subclasses of S are well-quasi-ordered by $\preceq$? For instance, Fellows et al. proved in [7] that graphs with bounded feedback-vertex-set are well-quasi-ordered by topological minors. Other papers considering this question include [1,3–6,8,9,13,15].

One way to approach this problem is to consider graph classes defined by excluded substructures. In this direction, Damaschke proved in [4] that a class of graphs defined by one forbidden induced subgraph H is wqo by the induced subgraph relation iff H is the path on four vertices. Similarly, a bit later Ding proved in [5] an analogous result for the subgraph relation. Other authors also considered this problem (see for instance [2,10,11]). In this paper, we provide the answer to the same question for the induced minor relation, which we denote $\leq_{\mathrm{im}}$. Before stating our main result, let us introduce two graphs which play a major role in this paper: $\hat{K}_4$ is obtained by adding a vertex of degree two to K_4 and the gem by adding a dominating vertex to P_4. (cf. Figure 1).

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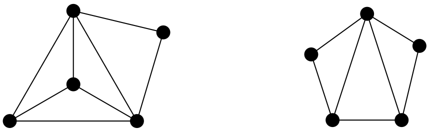

Fig. 1. The graph $\hat{K}_4$ (on the left) and the gem (on the right).

2 Induced minors and well-quasi-ordering

Our main result is the following.

Theorem 2.1 (Dichotomy Theorem) *Let H be a graph. The class of H-induced minor-free graphs is wqo by $\leq_{\text{im}}$ iff $H \leq_{\text{im}} \hat{K}_4$ or $H \leq_{\text{im}}$* gem.

Our proof naturally has two parts: for different values of H, we need to show wqo of H-induced minor-free graphs or exhibit an H-induced minor-free infinite antichain. Due to space limitations, we only present the main ideas of the proof of the dichotomy theorem.

2.1 Classes that are wqo

The following two theorems describe the structure of graphs with H forbidden as an induced minor, when H is $\hat{K}_4$ and the gem, respectively.

Theorem 2.2 (Decomposition of $\hat{K}_4$-induced minor-free graphs) *Let G be a 2-connected graph of* $\text{Excl}_{\text{im}}(\hat{K}_4)$. *Then:*

- *either $G \not\leq_{\text{im}} K_4$;*
- *or G is a subdivision of a graph on at most 9 vertices;*
- *or $\text{V}(G)$ has a partition (C, M) such that $G[C]$ is an induced cycle, $G[M]$ is a complete multipartite graph and every vertex of C is either adjacent in G to all vertices of M, or to none of them.*

Theorem 2.3 (Decomposition of gem**-induced minor-free graph)** *Let G be a 2-connected* gem*-induced minor-free graph. Then G has a subset $X \subseteq \text{V}(G)$ of at most six vertices such that every connected component of $G \setminus X$ is either a cograph, or a path whose internal vertices are of degree two in G.*

Using these structural results, we are able to show the wqo of the two classes with respect to induced minors.

2.2 Classes that are not wqo

For classes not covered by previous subsection, that is for any graph H which is not an induced minor of one of $\hat{K}_4$ and gem, we need to show that the H-induced minor-free graphs are not wqo by $\leq_{\mathrm{im}}$. The idea is to consider an infinite antichain for induced minors, and to show that infinitely many of its elements are H-induced minor-free. Let $\overline{G}$ denote the complement of any graph G. Using the infinite antichain $\{\overline{C_n}\}_{n\geq 6}$, we are able to prove the following lemma.

Lemma 2.4 *If the class of H-induced minor-free graphs is wqo by $\leq_{\mathrm{im}}$, then $\overline{H}$ is disjoint union of paths.*

Using Lemma 2.4 and two antichains introduced in [6] and in [12], we can deduce the following properties of $\overline{H}$. (Let $\mathrm{cc}(G)$ denote the number of connected components of a graph G.)

Lemma 2.5 *If H-induced minor-free graphs are wqo by $\leq_{\mathrm{im}}$, then (a) $\overline{H}$ has at most 4 connected components; (b) the largest connected component of $\overline{H}$ has at most 4 vertices; (c) if $\mathrm{cc}(\overline{H}) = 3$ then $|\mathrm{V}(H)| \leq 5$; and (d) if $\mathrm{cc}(\overline{H}) = 4$ then $|\mathrm{V}(H)| \leq 4$.*

Table 1 enumerates all the possible cases for $\overline{H}$, where each line corresponds to a fixed number of vertices and each column to a fixed value of $\mathrm{cc}(\overline{H})$. A gray cell means that either that no such graph exists, or that the cell corresponds to cases where H-induced minor-free graphs are not wqo by $\leq_{\mathrm{im}}$ (according to Lemma 2.5). The complement of any of the twelve remaining graphs can easily be shown to be induced minor of $\hat{K}_4$ or gem.

$\lvert\mathrm{V}(H)\rvert \setminus \mathrm{cc}(\overline{H})$	1	2	3	4	≥ 5
1	K_1				(a)
2	K_2	$2 \cdot K_1$			(a)
3	P_3	$K_2 + K_1$	$3 \cdot K_1$		(a)
4	P_4	$P_3 + K_1$	$K_2 + 2 \cdot K_1$	$4 \cdot K_1$	(a)
5	(b)	$P_4 + K_1$	$P_3 + 2 \cdot K_1$	(d)	(a)
≥ 6	(b)	(b)	(c)	(d)	(a)

Table 1
If H-induced minors-free graphs are wqo by $\leq_{\mathrm{im}}$, then $\overline{H}$ belongs to this table.

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