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Almost-spanning universality in random graphs (Extended abstract)

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Abstract

A graph G is said to be $\mathcal{H}(n, \Delta)$-universal if it contains every graph on n vertices with maximum degree at most Δ. It is known that for any $\varepsilon > 0$ and any natural number Δ there exists $c > 0$ such that the random graph $G(n, p)$ is asymptotically almost surely $\mathcal{H}((1-\varepsilon)n, \Delta)$-universal for $p \geq c(\log n/n)^{1/\Delta}$. Bypassing this natural boundary, we show that for $\Delta \geq 3$ the same conclusion holds when $p = \omega\left(n^{-\frac{1}{\Delta-1}} \log^5 n\right)$.

Keywords: Random graphs, Universality, Bounded-degree graphs

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1 Introduction

Given a family of graphs $\mathcal{H}$, a graph G is said to be $\mathcal{H}$-universal if it contains every member of $\mathcal{H}$ as a subgraph (not necessarily induced). Universal graphs have been studied quite extensively, particularly with respect to families of forests, planar graphs and graphs of bounded degree (see, for example, [4, 5, 7–12, 14] and their references). In particular, it is of interest to find sparse universal graphs.

Let $\mathcal{H}(n, \Delta)$ be the family of all graphs on n vertices with maximum degree at most Δ. Building on earlier work with several authors [2, 5, 6], Alon and Capalbo [3,4] showed that there are graphs with at most $c_\Delta n^{2-2/\Delta}$ edges which are $\mathcal{H}(n, \Delta)$-universal. A simple counting argument shows that this result is best possible.

The construction of Alon and Capalbo is explicit. An earlier approach had been to study whether random graphs could be $\mathcal{H}(n, \Delta)$-universal. The binomial random graph $G(n, p)$ is the graph formed by choosing every edge of a graph on n vertices independently with probability p. We say that $G(n, p)$ satisfies a property $\mathcal{P}$ asymptotically almost surely (a.a.s.) if $\Pr(G(n, p) \in \mathcal{P})$ tends to 1 as n tends to infinity. The first result on universality in random graphs was proved by Alon, Capalbo, Kohayakawa, Rödl, Ruciński and Szemerédi [5], who showed that for any $\varepsilon > 0$ there exists a constant $c > 0$ such that the random graph $G(n, p)$ is a.a.s. $\mathcal{H}((1 - \varepsilon)n, \Delta)$-universal for $p \geq c(\log n/n)^{1/\Delta}$.

Here we make some initial progress on improving the theorem of Alon et. al [5] on almost-spanning universality in random graphs.

Theorem 1.1 *For any constant $\varepsilon > 0$ and integer $\Delta \geq 3$, the random graph $G(n, p)$ is a.a.s. universal for the family $\mathcal{H}((1 - \varepsilon)n, \Delta)$, provided that $p = \omega\left(n^{-\frac{1}{\Delta-1}} \log^5 n\right)$.*

This result bypasses a natural barrier, since $(\log n/n)^{1/\Delta}$ is roughly the lowest probability at which we can expect that every collection of Δ vertices will have many neighbors in common, a condition which is extremely useful if one wishes to embed graphs of maximum degree Δ. On the other hand, the lowest probability at which one might hope that the random graph $G(n, p)$ is a.a.s. $\mathcal{H}((1 - \varepsilon)n, \Delta)$-universal is $n^{-2/(\Delta+1)}$. Indeed, below this probability, $G(n, p)$ will typically not contain $(1 - \varepsilon)\frac{n}{\Delta+1}$ vertex-disjoint copies of $K_{\Delta+1}$ (see, for example, [13]). Thus, for $\Delta = 3$ our result is optimal up to the logarithmic factor, while for $\Delta \geq 4$ the gap remains.

In proving Theorem 1.1, we will make use of a recent result of Ferber, Nenadov and Peter [12] which improves the bounds in [11] for families $\mathcal{H}' \subseteq \mathcal{H}(n, \Delta)$ of graphs which have no "dense" subgraphs, i.e. no subgraphs of density $\Delta/2$. When embedding a graph $H \in \mathcal{H}((1-\varepsilon)n, \Delta)$, we will first find a subgraph $H' \in \mathcal{H}'$ by removing all small components and certain short cycles in H. We then use the main result of [12] to embed H', after which we replace the short cycles and small components to find an embedding of H.

1.1 Notation

For a graph $G = (V, E)$, we denote by $v(G)$ and $e(G)$ the size of the vertex and edge sets, respectively. For a vertex $v \in V$, we write $\Gamma_G^{(i)}(v) := \{w \in V : \mathrm{dist}(v, w) = i\}$ for the set of vertices at distance exactly i from v. For simplicity, we let $\Gamma_G^{(0)}(v) := \{v\}$ and $\Gamma_G(v) := \Gamma_G^{(1)}(v)$. Furthermore, for a set $S \subseteq V$, we define $\Gamma_G^{(i)}(S) := \{w \in V : \min_{v \in S} \mathrm{dist}(v, w) = i\}$. Similarly, we let $B_G^{(i)}(v) := \bigcup_{j=0}^{i} \Gamma_G^{(j)}(v)$ be the *ball* of radius i around v in G, i.e., the set of all vertices at distance at most i from v. For an integer k and a set of vertices $S \subseteq V$, we say that S is k-independent if $B_G^{(k)}(v) \cap (S \setminus \{v\}) = \emptyset$, i.e., every two vertices in S are at distance at least $k+1$. If there is no risk of ambiguity, we omit G from the subscript.

2 Tools and preliminaries

In this section, we present some tools to be used in the proof of our main result.

2.1 Universality for some special classes of graphs

In the following definition, we introduce a family of graphs that admit a "nice partition".

Definition 2.1 Let n, d and t be positive integers and let ε be a positive number. The family of graphs $\mathcal{F}(n, t, \varepsilon, d)$ consists of all graphs H on n vertices for which there exists a partition $W_0, \ldots, W_t$ of $V(H)$ such that

(i) $|W_t| = \lfloor \varepsilon n \rfloor$,

(ii) $W_0 = \Gamma(W_t)$,

(iii) W_t is 3-independent,

(iv) W_i is 2-independent for every $1 \leq i \leq t-1$, and

(v) for every $1 \leq i \leq t$ and for every $w \in W_i$, w has at most d neighbors in $W_0 \cup \ldots \cup W_{i-1}$.

The following result, due to Ferber, Nenadov and Peter [12], shows that for an appropriate p a typical $G \sim G(n,p)$ is $\mathcal{F}(n,t,\varepsilon,d)$-universal.

Theorem 2.2 (Theorem 4.1 in [12]) *Let n and t be positive integers, let $d = d(n) \geq 2$ be an integer and let $\varepsilon < 1/(2d)$. Then the random graph $G(n,p)$ is a.a.s. $\mathcal{F}(n,t,\varepsilon,d)$-universal, provided that $p = \omega\left(\varepsilon^{-1} t n^{-1/d} \log^2 n\right)$.*

The following lemma will allow us to ignore small connected components in the proof of Theorem 1.1.

Lemma 2.3 *Let $\varepsilon > 0$ be a constant and $\Delta \geq 3$ and k integers. Then, for $p = \omega(n^{-2/(\Delta+1)})$, $G \sim G(n,p)$ a.a.s. has the following property: for every $V' \subseteq V(G)$ of order $|V'| \geq \varepsilon n$, $G[V']$ contains all connected graphs $H \in \mathcal{H}(\log^k n, \Delta)$.*

2.2 Systems of disjoint representatives in hypergraphs

The following lemma will allow us to remove and replace a set of short cycles in the proof of Theorem 1.1 (see Phase II in the proof of Theorem 1.1). We make no effort to optimize the logarithmic factor in the bound on the edge probability p.

Lemma 2.4 *Let $\varepsilon > 0$ be a constant, $\Delta \geq 3$, $3 \leq g \leq 2\log n$ and $t \leq \varepsilon n / 8\log^3 n$ be integers and let $D \subseteq [n]$ be a subset of order $|D| = \varepsilon n / \log n$. Then $G \sim G(n,p)$ satisfies the following with probability at least $1 - o(1/n)$, provided that $p \gg \left(\log^7 n / n\right)^{1/(\Delta-1)}$: for any family of subsets $\{W_{i,j}\}_{(i,j)\in[t]\times[g]}$, where*

(i) $W_{i,j} \subseteq V(G) \setminus D$ and $|W_{i,j}| = \Delta - 2$ for all $(i,j) \in [t] \times [g]$, and
(ii) $W_{i,j} \cap W_{i',j'} = \emptyset$ for all $i \neq i'$,

there exists a family of cycles $\{C_i = (c_{i_1}, \ldots, c_{i_g})\}_{i\in[t]}$, each of length g, such that

(i) $V(C_i) \subseteq G[D]$ and $V(C_i) \cap V(C_{i'}) = \emptyset$, for all $i \neq i'$, and
(ii) $W_{i,j} \subseteq \Gamma_G(c_{i_j})$ for all $(i,j) \in [t] \times [g]$.

Lemma 2.4 follows from the generalization of Hall's matching criterion due to Aharoni and Haxell [1].

Theorem 2.5 (Corollary 1.2, [1]) *Let g be a positive integer and $\mathcal{H} = \{H_1, \ldots, H_t\}$ a family of g-uniform hypergraphs on the same vertex set. If, for every $\mathcal{I} \subseteq [t]$, the hypergraph $\bigcup_{i\in\mathcal{I}} H_i$ contains a matching of size greater* *than $g(|\mathcal{I}|-1)$, then there exists a function $f:[t]\to\bigcup_{i=1}^t E(H_i)$ such that $f(i)\in E(H_i)$ and $f(i)\cap f(j)=\emptyset$ for $i\neq j$.*

3 Proof of Theorem 1.1

Our proof strategy goes as follows. Given a graph $H\in\mathcal{H}((1-\varepsilon)n,\Delta)$, we first remove small connected components from H, writing H_1 for the resulting graph. Working in H_1, we then remove some carefully chosen induced cycles of length at most $2\log n$ in such a way that the resulting graph H_2 belongs to the family of graphs $\mathcal{F}((1-\varepsilon')n,\Theta(\log^3 n),\varepsilon'',\Delta-1)$, for some parameters ε' and ε'' tending to zero with ε. Now, using Theorem 2.2, we find an embedding of H_2. Then, using Lemma 2.4, we replace the removed cycles. Finally, using Lemma 2.3, we complete the embedding of H by embedding small components one by one. We will now give a formal description of this procedure.

Preparing the graph G. Fix some $\varepsilon>0$ and integer $\Delta\geq 3$. Let $R,D_3,\dots,D_{2\log n}\subseteq[n]$ be arbitrarily chosen disjoint subsets of $\{1,\dots,n\}$ such that $|R|=(1-\varepsilon/2)n$ and $|D_i|=\varepsilon n/4\log n$ for each $i\in\{3,\dots,2\log n\}$. Let G be a graph with the following properties:

(i) the induced subgraph $G[R]$ is $\mathcal{F}((1-\varepsilon/2)n,(\Delta^2+1)q+1,\varepsilon',\Delta-1)$-universal, where $q=65\varepsilon^{-1}\log^3 n$ and $\varepsilon'=\min\{1/2\Delta,\varepsilon/(2-\varepsilon)\}$,

(ii) for every subset $V'\subseteq V(G)$ of order $|V'|\geq\varepsilon n$, the induced subgraph $G[V']$ contains every connected graph from the family $\mathcal{H}(\log^4 n,\Delta)$, and

(iii) G satisfies the property given by Lemma 2.4 for every $g\in\{3,\dots,2\log n\}$, $t\leq\varepsilon n/(32\log^3 n)$ and $D=D_g$.

Observe that by Theorem 2.2 and Lemmas 2.3 and 2.4, the random graph $G\sim G(n,p)$ satisfies properties (i)-(iii) asymptotically almost surely, provided that $p=\omega\left(n^{-1/(\Delta-1)}\log^5 n\right)$. We remark that the bound on p here is determined by Theorem 2.2.

Preparing the graph H. Let $H\in\mathcal{H}((1-\varepsilon)n,\Delta)$ and let $H_1\subseteq H$ be the subgraph which consists of all connected components of H with at least $\log^4 n$ vertices. Moreover, let $I\subseteq V(H_1)$ be a maximal $(64\varepsilon^{-1}\log^3 n)$-independent set in H_1. It is not difficult to see that $|I|\leq\frac{\varepsilon n}{32\log^3 n}$. The following observation plays a crucial role in our argument.

Claim 3.1 *For every vertex $v\in I$ at least one of the following properties hold:*

(a) $B_{H_1}^{(\log n)}(v)$ contains a vertex w with $\deg_{H_1}(w)\leq\Delta-1$, or

(b) $H_1[B_{H_1}^{(\log n)}(v)]$ *contains a cycle of length at most* $2\log n$.

Proof. Let us assume the opposite, i.e., for every vertex $w \in B_{H_1}^{(\log n)}(v)$ we have $\deg_{H_1}(w) = \Delta$ and $H_1[B_{H_1}^{(\log n)}(v)]$ contains no cycle of length at most $2\log n$. Then $H_1[B_{H_1}^{(\log n)}(v)]$ is a tree and, since $\Delta \geq 3$, it contains at least $\sum_{j=1}^{\log n}(\Delta-1)^j > n$ vertices, which is clearly a contradiction. □

Let us write I_a for the set of all vertices in I which satisfy property (a) of Claim 3.1 and set $I_b := I \setminus I_a$. Furthermore, for each $v \in I_b$, let C_v be a cycle of smallest length in $H_1[B_{H_1}^{(\log n)}(v)]$, let ℓ_v denote its length and fix an arbitrary ordering $(c_v^1, \ldots, c_v^{\ell_v})$ of the vertices along C_v. By minimality, C_v is an induced cycle. Finally, let $H_2 := H_1 \setminus \left[\bigcup_{v\in I_b} V(C_v)\right]$ and note that

$$B_{H_1}^{(3\log n)}(v) \cap V(H_2) = \begin{cases} B_{H_1}^{(3\log n)}(v), & \text{for } v \in I_a, \\ B_{H_1}^{(3\log n)}(v) \setminus V(C_v), & \text{for } v \in I_b. \end{cases} \tag{1}$$

Phase I: Embedding H_2 into $G[R]$. We claim that there exists an embedding of H_2 into $G[R]$. Let H_2' be a graph on $(1-\varepsilon/2)n$ vertices obtained from H_2 by adding isolated vertices. Using property (i) of the graph G, in order to show that there exists an embedding of H_2 into $G[R]$, it will suffice to prove that $H_2' \in \mathcal{F}((1-\varepsilon/2)n, (\Delta^2+1)q+1, \varepsilon', \Delta-1)$, where $q = 65\varepsilon^{-1}\log^3 n$ and $\varepsilon' = \min\{1/2\Delta, \varepsilon/(2-\varepsilon)\}$. We prove this by finding a partition $W_0, W_1, \ldots, W_{(\Delta^2+1)q+1}$ of $V(H_2')$ with the following properties:

(i) $|W_{(\Delta^2+1)q+1}| = \lfloor \varepsilon'(1-\varepsilon/2)n \rfloor$,

(ii) $W_0 = \Gamma_{H_2'}(W_{(\Delta^2+1)q+1})$,

(iii) $W_{(\Delta^2+1)q+1}$ is 3-independent (in H_2'),

(iv) W_i is 2-independent (in H_2') for every $1 \leq i \leq (\Delta^2+1)q$, and

(v) for every $1 \leq i \leq (\Delta^2+1)q+1$ and for every $w \in W_i$, w has at most $\Delta-1$ neighbors in $W_0 \cup \ldots \cup W_{i-1}$.

First, note that H_2' contains at least $\varepsilon n/2$ isolated vertices as $|V(H_2)| \leq (1-\varepsilon)n$. Since $\varepsilon' \leq \varepsilon/(2-\varepsilon)$ or equivalently $\varepsilon'(1-\varepsilon/2) < \varepsilon/2$, we can set $W_{(\Delta^2+1)q+1}$ to be a set of $\varepsilon'(1-\varepsilon/2)n$ isolated vertices. Then $W_0 = \emptyset$ and $W_{(\Delta^2+1)q+1}$ is trivially 3-independent. Furthermore, let $S_q \subseteq V(H_2') \setminus W_{(\Delta^2+1)q+1}$ be the set of all remaining vertices in H_2' with degree at most $\Delta-1$ and observe that for each $v \in I$, we have

$$S_q \cap B_{H_1}^{(3\log n)}(v) \neq \emptyset. \tag{2}$$

For $v \in I_a$, this follows from (1) and the definition of the set I_a. When $v \in I_b$, we have from (1) and $|B_{H_1}^{(3\log n)}(v)| \geq 3\log n > |V(C_v)|$ that $B_{H_1}^{(3\log n)}(v) \cap V(H_2) \neq \emptyset$. Thus there exists a vertex $w \in B_{H_1}^{(3\log n)}(v) \cap V(H_2)$ adjacent to some vertex in C_v, and clearly $\deg_{H_2}(w) \leq \Delta - 1$.

Next, for each $i \in \{1, \ldots, q-1\}$, we define

$$S_{q-i} := \Gamma_{H_2}^{(i)}(S_q).$$

We first show that $S_1, \ldots, S_q, W_{(\Delta^2+1)q+1}$ is a partition of $V(H_2')$. Since disjointness follows from the construction, it suffices to prove that for each $w \in V(H_2') \setminus W_{(\Delta^2+1)q+1}$ we have $B_{H_2}^{(q-1)}(w) \cap S_q \neq \emptyset$. This can be seen as follows. Since I is a maximal $(64\varepsilon^{-1}\log^3 n)$-independent set in H_1, for each vertex $w \in V(H_2) \setminus I$ we have $B_{H_1}^{(64\varepsilon^{-1}\log^3 n)}(w) \cap I \neq \emptyset$. Otherwise, we could extend I, contradicting its maximality. Thus, from (2) we conclude that for each $w \in V(H_2) \setminus I$ we have $B_{H_1}^{(q-1)}(w) \cap S_q \neq \emptyset$. Let us now consider the shortest path in H_1 from w to a vertex $s \in S_q$, and denote the vertices along such a path as $w = p_0, p_1, p_2, \ldots, p_{q'} = s$, for some $q' \leq q-1$. If all vertices $p_1, \ldots, p_{q'} \in V(H_2)$, then clearly $s \in B_{H_2}^{q-1}(w)$. Otherwise, let i' be the smallest index such that $p_{i'} \notin V(H_2)$. But then $\deg_{H_2}(p_{i'-1}) \leq \Delta - 1$ and thus by the definition $p_{i'-1} \in S_q$, which implies $B_{H_2}^{q-1}(w) \cap S_q \neq \emptyset$. This shows that $S_1, \ldots, S_q, W_{(\Delta^2+1)q+1}$ is indeed a partition of $V(H_2')$.

Furthermore, by construction, for each $i \in \{1, \ldots, q-1\}$ and each vertex $v \in S_i$, v has at least one neighbor in $\bigcup_{j=i+1}^{q} S_j$ and thus at most $\Delta - 1$ neighbors in $\bigcup_{j=0}^{i-1} S_j$. However, the sets S_i are not necessarily 2-independent in H_2'. This can be fixed in the following way. The square of H_2', denoted by $(H_2')^2$, has maximum degree at most Δ^2. Therefore, $(H_2')^2$ can be partitioned into $\Delta^2 + 1$ sets $L_1, \ldots, L_{\Delta^2+1}$ which are independent in $(H_2')^2$ and thus 2-independent in H_2'. Now, by setting $W_{(i-1)(\Delta^2+1)+j} := S_i \cap L_j$ for every $i \in \{1, \ldots, q\}$ and $j \in \{1, \ldots, \Delta^2+1\}$, we obtain a partition of $V(H_2')$ satisfying properties (i)–(v).

To conclude, we have shown that $H_2' \in \mathcal{F}((1-\varepsilon/2)n, (\Delta^2+1)q+1, \varepsilon', \Delta-1)$. Thus, by property (i) of the graph G, there exists an embedding $f : V(H_2) \to R$ of H_2 into $G[R]$.

Phase II: Embedding removed cycles. Consider some $g \in \{3, \ldots, 2\log n\}$ and let $I_g \subseteq I_b$ be the set of all vertices $v \in I_b$ such that $\ell_v = g$. For each $(v, j) \in I_g \times [g]$, let $W_{v,j} := f(\Gamma_H(c_v^j) \cap V(H_2))$ and observe that such family of subsets satisfies the requirements of Lemma 2.4 with $D = D_g$. Therefore, by property (iii) of the graph G, there exists a family $\{(c_{v,1}, \ldots, c_{v,g})\}_{v \in I_g}$

of vertex disjoint cycles in $G[D_g]$ such that setting $f(c_v^j) := c_{v,j}$ for every $(v,j) \in I_g \times [g]$ defines an embedding of $H_2 \cup \left[\bigcup_{v\in I_g} C_v\right]$ into $G[R \cup D_g]$. Since this holds for every $3 \leq g \leq 2\log n$ and the sets $D_3, \ldots, D_{2\log n}$ are disjoint, we obtain an embedding of H_1 into G.

Phase III: Embedding small components. As a last step, we have to extend our embedding of H_1 to an embedding of the whole graph H. Using the facts that H is of order $(1-\varepsilon)n$ and that each component of H which is not in H_1 is of order at most $\log^4 n$, we can greedily embed these components one by one as follows. Consider one such component and let $V' \subseteq V(G)$ be the set of vertices which are not an image of some already embedded vertex of H. Then $|V'| \geq \varepsilon n$ and by property (ii) of the graph G, $G[V']$ contains an embedding of the required component. Repeating the same argument for each component which has not yet been embedded, we obtain an embedding of the graph H.

4 Concluding remarks

The observant reader will have noticed that our argument does not apply when $\Delta = 2$. In this case, one cannot hope to show that universality holds all the way down to $p \approx n^{-1/(\Delta-1)} = n^{-1}$. Even to find a collection of $(1-\varepsilon)\frac{n}{3}$ disjoint triangles, the probability must be at least $n^{-2/3}$. Since every graph with maximum degree 2 is a disjoint union of paths and cycles, it is not too hard to use arguments similar to those of Section 2.1 to show that for any $\varepsilon > 0$ there exists a constant $c > 0$ such that if $p \geq cn^{-2/3}$, the random graph $G(n,p)$ is a.a.s. $\mathcal{H}((1-\varepsilon)n, 2)$-universal.

Our proof relies heavily on the fact that the graphs we are hoping to embed are almost spanning rather than spanning. In particular, we neither know how to complete the removed cycles nor how to add small components back into the graph without making heavy use of the almost spanning condition. Given this, it seems likely that a spanning analogue of Theorem 1.1 will require new ideas.

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