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On the zone of a circle in an arrangement of lines

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Abstract

Let $\mathcal{L}$ be a set of n lines in the plane, and let C be a convex curve in the plane, like a circle or a parabola. The *zone* of C in $\mathcal{L}$, denoted $\mathcal{Z}(C,\mathcal{L})$, is defined as the set of all faces in the arrangement $\mathcal{A}(\mathcal{L})$ that are intersected by C. Edelsbrunner et al. (1992) showed that the complexity (total number of edges or vertices) of $\mathcal{Z}(C,\mathcal{L})$ is at most $O(n\alpha(n))$, where α is the inverse Ackermann function, by translating the sequence of edges of $\mathcal{Z}(C,\mathcal{L})$ into a sequence S that avoids the subsequence *ababa*. Whether the worst-case complexity of $\mathcal{Z}(C,\mathcal{L})$ is only linear is a longstanding open problem.

In this paper we provide evidence that, if C is a circle or a parabola, then the zone of C has at most linear complexity: We show that a certain configuration of segments with endpoints on C is impossible. As a consequence, the Hart–Sharir sequences, which are essentially the only known way to construct *ababa*-free sequences of superlinear length, cannot occur in S.

Hence, if it could be shown that every family of superlinear-length, *ababa*-free sequences must eventually contain all Hart–Sharir sequences, that would settle the zone problem for a circle/parabola.

Keywords: arrangement, zone, Davenport–Schinzel sequence, inverse Ackermann function

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1 Introduction

Let $\mathcal{L}$ be a set of n lines in the plane. The *arrangement* of $\mathcal{L}$, denoted $\mathcal{A}(\mathcal{L})$, is the partition of the plane into vertices, edges, and faces induced by $\mathcal{L}$. Let C be another object in the plane. The *zone* of C in $\mathcal{L}$, denoted $\mathcal{Z}(C, \mathcal{L})$, is defined as the set of all faces in $\mathcal{A}(\mathcal{L})$ that are intersected by C. The *complexity* of $\mathcal{Z}(C, \mathcal{L})$ is defined as the total number of edges, or vertices, in it.

The celebrated *zone theorem* states that, if C is another line, then $\mathcal{Z}(C, \mathcal{L})$ has complexity $O(n)$ (Chazelle et al. [3]; see also Edelsbrunner et al. [5], Matoušek [12]).

If C is a convex curve, like a circle or a parabola, then $\mathcal{Z}(C, \mathcal{L})$ is known to have complexity $O(n\alpha(n))$, where α is the very-slow-growing inverse Ackermann function (Edelsbrunner et al. [5]; see also Bern et al. [2], Sharir and Agarwal [21]). More specifically, the *outer zone* of $\mathcal{Z}(C, \mathcal{L})$ (the part that lies outside the convex hull of C) is known to have complexity $O(n)$, whereas the complexity of the *inner zone* is only known to be $O(n\alpha(n))$. Whether the complexity of the inner zone is linear as well is a longstanding open problem [2,21].

In this paper we make progress towards proving that the inner zone of a circle, or a parabola, in an arrangement of lines has linear complexity. The problem is more naturally formulated with a circle, but a parabola is easier to work with. Therefore, throughout most of this paper we will take for concreteness C to be the parabola $y = x^2$. In the full version of this paper we show how to modify our argument for the case of a circle.

1.1 Davenport–Schinzel sequences and their generalizations

Let S be a finite sequence of symbols, and let $s \geq 1$ be a parameter. Then S is called a *Davenport–Schinzel sequence of order s* if every two adjacent symbols in S are distinct, and if S does not contain any alternation $a \cdots b \cdots a \cdots b \cdots$ of length $s+2$ for two distinct symbols $a \neq b$. Hence, for $s = 1$ the "forbidden pattern" is *aba*, for $s = 2$ it is *abab*, for $s = 3$ it is *ababa*, and so on.

The maximum length of a Davenport–Schinzel sequence of order s that contains only n distinct symbols is denoted $\lambda_s(n)$. For $s \leq 2$ we have $\lambda_1(n) = n$ and $\lambda_2(n) = 2n - 1$. However, for fixed $s \geq 3$, $\lambda_s(n)$ is slightly superlinear in n.

The case $s = 3$ is the one most relevant to us. Hart and Sharir [7] constructed a family of sequences that achieve the lower bound [2] $\lambda_3(n) \geq$

[2] See [13] on how to avoid losing a factor of 2 in the interpolation step.

$n\alpha(n) - O(n)$; and they also proved the asymptotically matching upper bound $\lambda_3(n) \le O(n\alpha(n))$. Klazar [10] subsequently improved the upper bound to $\lambda_3(n) \le 2n\alpha(n) + O(n\sqrt{\alpha(n)})$ (recently, Pettie [18] improved the lower-order term to $O(n)$). Nivasch [13] showed that $\lambda_3(n) \ge 2n\alpha(n) - O(n)$ by extending the Hart–Sharir sequences. Hence, $\lambda_3(n) = 2n\alpha(n) \pm O(n)$.

For $s = 4$ we have $\lambda_4(n) = \Theta(n \cdot 2^{\alpha(n)})$, and in general, $\lambda_s(n) = \Theta\big(n \cdot 2^{\mathrm{poly}(\alpha(n))}\big)$ for fixed $s \ge 4$, where the polynomial in the exponent is of degree roughly $s/2$. See Sharir and Agarwal [21], and subsequent improvements by Nivasch [13], and Pettie [18].

A *generalized Davenport–Schinzel sequence* is one where the forbidden pattern is not restricted to be $abab\cdots$, but it can be any fixed subsequence u. In order for the problem to be nontrivial we must require S to be k-*sparse*—meaning, every k adjacent symbols in S must be pairwise distinct—where $k = \|u\|$ is the number of distinct symbols in u. For example, if we take $u = abcaccbc$, then S must not contain any subsequence of the form $a \cdots b \cdots c \cdots a \cdots c \cdots c \cdots b \cdots c$ for $|\{a, b, c\}| = 3$, and every three adjacent symbols in S must be pairwise distinct.

We denote by $\mathrm{Ex}(u, n)$ the maximum length of a k-sparse, u-avoiding sequence S on n distinct symbols, where $k = \|u\|$. For every fixed forbidden pattern u, $\mathrm{Ex}(u, n)$ is at most slightly superlinear in n: $\mathrm{Ex}(u, n) = O\big(n \cdot 2^{\mathrm{poly}(\alpha(n))}\big)$, where the polynomial in the exponent depends on u (Klazar [8], Nivasch [13], Pettie [19]).

Similarly, $\mathrm{Ex}(\{u_1, u_2, \ldots, u_j\}, n)$ denotes the maximum length of a sequence that avoids all the patterns $u_1, \ldots, u_j$, is k-sparse for $k = \min\{\|u_1\|, \ldots, \|u_j\|\}$, and contains only n distinct symbols.

Here we recall the following known facts:

- $\mathrm{Ex}(\{ababa, ab\,cac\,cbc\}, n) = \Theta(n\alpha(n))$ (Pettie [16]). Indeed, the $ababa$-free sequences of Hart and Sharir [7] avoid $ab\,cac\,cbc$ as well. [3]
- $\mathrm{Ex}(ab\,cacbc, n) = \Theta(n\alpha(n))$ (Pettie [17]). The lower bound is achieved by a modification of the Hart–Sharir construction, which does not avoid $ababa$ anymore.
- It is unknown whether $\mathrm{Ex}(\{ababa, ab\,cacbc\}, n)$ or $\mathrm{Ex}(\{ababa, ab\,cac\,cbc, (ab\,cac\,cbc)^R\}, n)$ are superlinear in n (where u^R denotes the reversal of u). We conjecture that they are both $O(n)$.

[3] Spaces are just for clarity.

1.2 Transcribing the zone into a Davenport–Schinzel sequence

Here we recall the argument of Edelsbrunner et al. [5] showing that the inner complexity of $\mathcal{Z}(C, \mathcal{L})$ is $O(n\alpha(n))$.

Let $\mathcal{L}$ be a set of n lines in the plane, and assume for simplicity that C is the parabola $y = x^2$. Also assume general position for simplicity.

The lines $\mathcal{L}$ partition the convex hull of C into faces, only one of which is unbounded. Let $\mathcal{L}'$ be the set of n segments obtained by intersecting each line of $\mathcal{L}$ with the convex hull of C.

The complexity of the unbounded face is at most n (as is the complexity of any single face). To bound the complexity of the remaining faces, we traverse the boundary of the inner zone by starting at the leftmost endpoint of $\mathcal{L}'$, and walking around the boundary of the faces, as if the segments were walls which we touch with the left hand at all times, until we reach the rightmost endpoint of $\mathcal{L}'$. See Figure 1. We transcribe this tour into a sequence containing $3n$ distinct symbols as follows:

Each segment $a \in \mathcal{L}'$ is partitioned by the other segments into smaller pieces. We take two directed copies of each such piece. We call each such copy a *sub-segment*. One sub-segment is placed slightly above a and is directed leftwards, and the other one is placed slightly below a and is directed rightwards. Hence, our tour visits some of these sub-segments, in the directions we have given them, in a certain order.

For each segment a, the sub-segments of a that are visited, are visited in counterclockwise order around a. We first visit some upper sub-segments from right to left, then we visit some lower sub-segments from left to right, and then we again visit some upper sub-segments from right to left.

Sub-segments of the first type are transcribed as a'; sub-segments of the second type are transcribed as a, and sub-segments of the third type are transcribed as a''. See again Figure 1. Let S' be the sequence resulting from the tour.

The restriction of S' to first-type symbols contains no alternation *abab*, and it contains no adjacent repetitions either. Hence, it is an order-2 DS-sequence and so it has linear length. The same is true for the restriction of S' to third-type symbols.

Thus, the important part of the sequence S' is its restriction to second-type symbols. From now on we denote this subsequence S, and we call it the *lower inner-zone sequence of* $\mathcal{Z}(C, \mathcal{L})$. The sequence S contains no alternation *ababa*, and it contains no adjacent repetitions. Hence, S is an order-3 DS-sequence, and hence its length is at most $O(n\alpha(n))$.

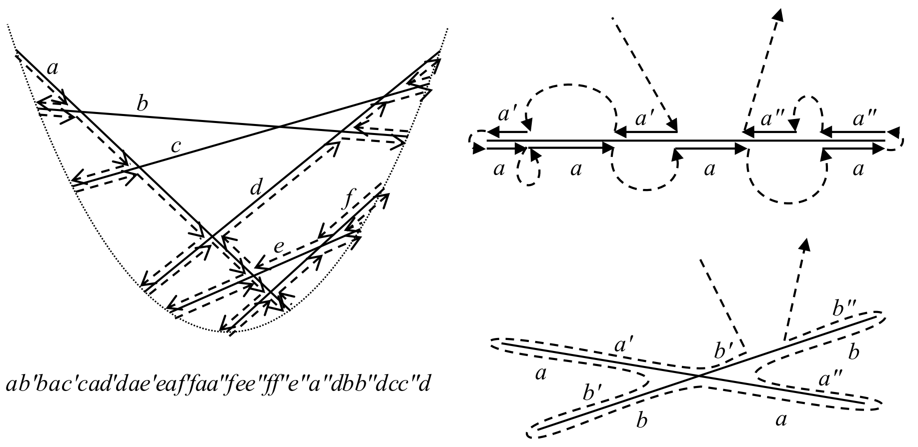

Fig. 1. Traversing the boundary of the inner zone of the parabola.

1.3 Our results

In this paper we offer some evidence for the following conjecture, and make some progress towards proving it:

Conjecture 1.1 *If $\mathcal{L}$ is a set of n lines and C is a circle or a parabola, then the lower inner-zone sequence S of $\mathcal{Z}(C, \mathcal{L})$ has length $O(n)$, and hence $\mathcal{Z}(C, \mathcal{L})$ has at most linear complexity.*

Our main result is that there exists a Hart–Sharir sequence that cannot occur as a subsequence of S.

We first present in Section 3 a certain, relatively simple configuration of eleven segments that is impossible. It follows that the sequence S must avoid a certain pattern u of length 33. This result, however, is useless for establishing Conjecture 1.1, since u contains both $ab\,cac\,cbc$ and its reversal. Therefore, by the above-mentioned result of Pettie, the Hart–Sharir construction avoids both u and u^R (which is actually the same as u), and so $\mathrm{Ex}(\{ababa, u, u^R\}, n) = \Theta(n\alpha(n))$.

Section 3 is just a warmup for Section 4. There we construct another impossible configuration X, this time with 173 segments. We could construct a pattern u' that, as a consequence, cannot occur in S, but we abstain from doing so. Instead, we show in the full version of this paper that the Hart–Sharir sequences eventually force the configuration X.

Finally, in Section 5 we discuss why we believe our results support Conjecture 1.1, and we suggest a possible line of attack. We conclude with some open problems.

2 Preliminaries

Throughout this and the following sections C will denote the parabola $y = x^2$, and $\mathcal{L}'$ will denote a set of n segments with endpoints on C.

The left and right endpoints of a segment $a \in \mathcal{L}'$ will be denoted L_a and R_a, respectively. Whenever we say that a sequence of endpoints appear in a certain order, we mean from left to right.

Two segments a, b intersect if and only if their endpoints appear in the order $L_a\, L_b\, R_a\, R_b$ or $L_b\, L_a\, R_b\, R_a$.

If $a_1, \ldots, a_m$ are segments whose endpoints appear in the order $L_{a_1} \cdots L_{a_m}$ $R_{a_1} \cdots R_{a_m}$, then they pairwise intersect. If the intersection points $a_m \cap a_{m-1}$, …, $a_3 \cap a_2$, $a_2 \cap a_1$ appear in this order from left to right, then we say that the segments *intersect concavely.* If the intersection points appear in the reverse order, then we say that the segments *intersect convexly.*

We will specify configurations of segments by listing the order of their endpoints, and by specifying that some subsets of segments must intersect concavely. We will prove that some configurations are geometrically impossible.

Lemma 2.1 *Let a, b, c, d be four points on the parabola C, having increasing x-coordinates $a_x < b_x < c_x < d_x$. Let $z = ac \cap bd$. Define the horizontal distances $p = b_x - a_x$, $q = d_x - c_x$, $r = z_x - b_x$, $s = c_x - z_x$. Then $p/q = r/s$.*

Observation 2.2 *Let S be the lower inner-zone sequence corresponding to $\mathcal{L}'$.*

(i) *If S contains the subsequence abab, then segments $a, b \in \mathcal{L}'$ cross, and their endpoints are ordered L_a, L_b, R_a, R_b from left to right.*

(ii) *If S contains the subsequence abcbc, then L_a lies left of L_c. Similarly, if S contains cbcba, then R_c lies left of R_a.*

(iii) *If S contains axaxbxb or axaxybyb, then R_a lies left of L_b.*

Observation 2.3 *If S contains the "N-shaped" subsequence $12 \cdots m \cdots 212 \cdots m$, then the corresponding segments must have endpoints in the order $L_1 \cdots L_m\, R_1 \cdots R_m$, and must intersect concavely.*

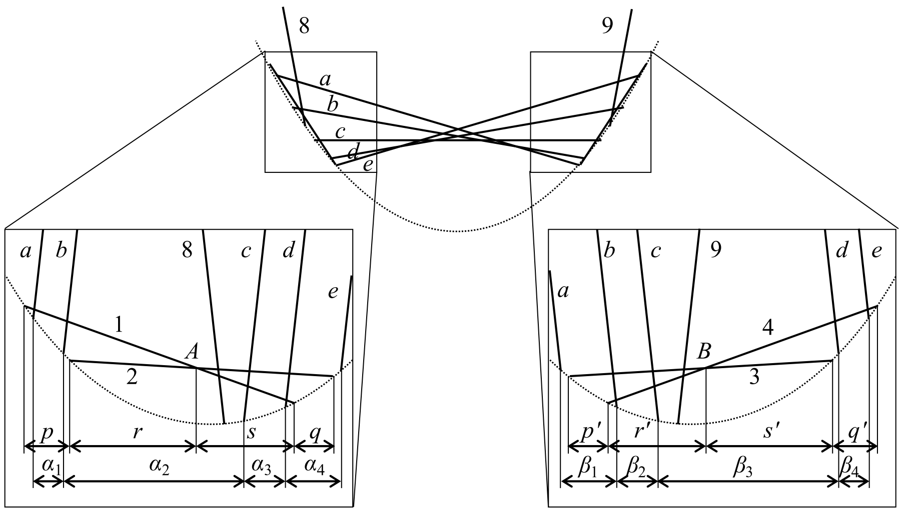

Fig. 2. An impossible configuration of segments.

3 Warmup: A simple but useless impossible configuration

Theorem 3.1 *Let $a, b, c, d, e, 1, 2, 3, 4, 8, 9$ be eleven segments with endpoints on the parabola C, in left-to-right order*

$$L_8\, L_1\, L_a\, L_b\, L_2\, R_8\, L_c\, L_d\, R_1\, R_2\, L_e\, R_a\, L_3\, L_4\, R_b\, R_c\, L_9\, R_3\, R_d\, R_e\, R_4\, R_9\,. \quad (1)$$

Then, it is impossible for segments $8, 1, 2$ to intersect concavely, segments $3, 4, 9$ to intersect concavely, and segments a, b, c, d, e to intersect concavely, all at the same time. See Figure 2.

Corollary 3.2 *Let S be the lower inner-zone sequence of the parabola C in an arrangement of lines. Then S cannot contain a subsequence isomorphic to*

$$u = 81ab12181cd12dedcbab34bc49434de49.$$

4 A more promising impossible configuration

We now consider endpoint sequences in which some contiguous subsequences (*blocks*) that contain only left endpoints are designated as *special blocks*. It will always be the case that all the special blocks in a sequence have the same length. We denote special blocks by enclosing them in parentheses.

We define an operation on endpoint sequences called *endpoint shuffling*. Let A be sequence that has k special blocks of length m, and let B be a sequence that has ℓ special blocks of length k. Then the *endpoint shuffle* of A and B, denoted $A \circ B$, is a new sequence having $k\ell$ special blocks of length $m+1$, formed as follows: We make ℓ copies of A (one for each special block of B), each one having "fresh" symbols that do not occur in B nor in any other copy of A.

For each special block $\Gamma_i = (L_1 \ldots L_k)$ in B, $1 \le i \le \ell$, let A_i be the i-th copy of A. We insert each L_j at the end of the j-th special block of A_i. Then we insert the resulting sequence in place of Γ_i in B. The result of all these replacements is the desired sequence $A \circ B$.

For example, let

$$A = (L_a)\,(L_b)\,(L_c)\,R_a\,R_b\,R_c, \qquad B = (L_1\,L_2\,L_3)\,(L_4\,L_5\,L_6)\,R_1\,R_4\,R_2\,R_5\,R_3\,R_6.$$

Then,

$$\begin{aligned} A \circ B = (L_a\,L_1)\,(L_b\,L_2)\,(L_c\,L_3)\,R_a\,R_b\,R_c\,(L_{a'}\,L_4) \\ (L_{b'}\,L_5)\,(L_{c'}\,L_6)\,R_{a'}\,R_{b'}\,R_{c'}\,R_1\,R_4\,R_2\,R_5\,R_3\,R_6. \end{aligned}$$

Now, define the following endpoint sequences:

$$\begin{aligned} F_m &= (L_1 \cdots L_m) \quad R_1 \cdots R_m, \qquad m \ge 1; \\ Z_m &= L_a\,L_b \quad (L_1 \cdots L_m) \quad R_1 \cdots R_m \quad L_c\,R_a \\ &\qquad (L_{m+1} \cdots L_{2m}) \quad R_{m+1} \cdots R_{2m} \quad R_b\,R_c, \qquad m \ge 1; \\ Y &= L_d\,L_e\,()\,()\,L_f\,R_d\,()\,()\,R_e\,R_f\,(), \end{aligned}$$

where Y has five empty special blocks.

Define the endpoint sequence

$$X = Y \circ \big((((Z_1 \circ Z_2) \circ Z_4) \circ Z_8) \circ F_{16}\big).$$

X contains 15 sets of segments of types a, b, c that come from 15 copies of Z_m; 16 sets of segments of types d, e, f that come from 16 copies of Y; and 80 "numeric" segments, which are partitioned into 16 5-tuples, according to the 16 copies of Y that contain their left endpoints. Hence, X contains a total of 173 segments.

Theorem 4.1 *It is impossible to realize X such that the segments a, b, c in each copy of Z_m intersect concavely, the segments d, e, f in each copy of Y*

intersect concavely, and the five numeric segments in each 5*-tuple intersect concavely.*

5 Discussion

We believe the Hart–Sharir sequences are the *only* way to achieve superlinear-length *ababa*-free sequences.[4] Specifically, we conjecture:

Conjecture 5.1 *For every Hart–Sharir sequence* $S_k(m)$ *we have*

$$\mathrm{Ex}\big(\{ababa, S_k(m), (S_k(m))^R\}, n\big) = O(n);$$

where the hidden constant depends on k *and* m.

Conjecture 5.1 implies Conjecture 1.1. Conjecture 5.1 is known to be true for $k = 1$, since $S_1(m)$ are N-shaped sequences [11,15]. Hence, the first open case is $S_2(2) = aba\,cdcac\,dbd$ (which is the same as $(S_2(2))^R$). However, as we mentioned in the Introduction, even the weaker conjecture, that $\mathrm{Ex}(\{ababa, ab\,cacbc\}, n) = O(n)$, is still open.

5.1 Related open problems

- What if we do not require C to be a parabola, but only a convex curve? It still seems impossible to implement the above-mentioned construction.
- The longest Davenport–Schinzel sequences of order 4 (*ababab*-free) have length $\Theta\big(n \cdot 2^{\alpha(n)}\big)$. However, no one knows how to realize them as lower-envelope sequences of parabolic segments. Perhaps it is impossible. One could start by finding forbidden patterns here as well.
- *Higher dimensions:* Raz [20] recently proved that the combinatorial complexity of the outer zone of the boundary of a convex body in an arrangement of hyperplanes in R^d is $O(n^{d-1})$. The complexity of the inner zone is only known to be $O(n^{d-1} \log n)$ (Aronov et al. [1]). Whether the latter is also linear in n is an open question.

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[4] The longer sequences of Nivasch [13] contain the Hart–Sharir sequences.

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