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Rainbow matchings and algebras of sets

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Abstract

Grinblat (2002) asks the following question in the context of algebras of sets: What is the smallest number $\mathfrak{v} = \mathfrak{v}(n)$ such that, if $A_1, \dots, A_n$ are n equivalence relations on a common finite ground set X, such that for each i there are at least $\mathfrak{v}$ elements of X that belong to A_i-equivalence classes of size larger than 1, then X has a rainbow matching—a set of $2n$ distinct elements $a_1, b_1, \dots, a_n, b_n$, such that a_i is A_i-equivalent to b_i for each i?

Grinblat has shown that $\mathfrak{v}(n) \le 10n/3 + O(\sqrt{n})$. He asks whether $\mathfrak{v}(n) = 3n - 2$ for all $n \ge 4$. In this paper we improve the upper bound (for all large enough n) to $\mathfrak{v}(n) \le 16n/5 + O(1)$.

Keywords: rainbow matching, algebra of sets, equivalence relation

1 Introduction

Let n be a positive integer. Let X be a finite "ground set", and let $A_1, \dots, A_n$ be n equivalence relations on X (or equivalently, partitions of X into subsets). If $a, b \in X$ are equivalent under A_i, then we say for short that a, b are i-equivalent, and we write $a \sim_i b$. The i-equivalence class of an element $a \in X$

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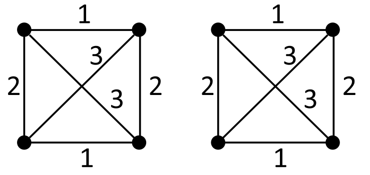

Fig. 1. Here $|K_i| = 8$ for all $i = 1, 2, 3$, and yet there is no rainbow matching.

is given by $[a]_i = \{b \in X : a \sim_i b\}$. The *kernel* of A_i, denoted K_i, is defined as the set of elements of X that are i-equivalent to some element other than themselves:

$$K_i = \{a \in X : |[a]_i| > 1\}.$$

(It will become evident that one can assume without loss of generality that all equivalence classes in each A_i have size at most 3.)

We shall call a set of $2n$ *distinct* elements $a_1, b_1, \ldots, a_n, b_n \in X$ a *rainbow matching* if $a_i \sim_i b_i$ for each i. (See e.g. Glebov et al. [1] for the term.)

Grinblat has studied this notion in the context of algebras of sets [2,3,4]. He asks for the minimum number $\mathfrak{v} = \mathfrak{v}(n)$ such that, if $|K_i| \geq \mathfrak{v}$ for all i, then $A_1, \ldots, A_n$ have a rainbow matching [2].

Grinblat observed that $\mathfrak{v}(n) \geq 3n-2$: If we let all equivalence relations A_i be identical, consisting of $n-1$ equivalence classes of size 3, then they have no rainbow matching even though $|K_i| = 3n-3$.

Grinblat also showed that $\mathfrak{v}(3) = 9$. The lower bound $\mathfrak{v}(3) > 8$ is illustrated in Figure 1.

Grinblat recently proved that $\mathfrak{v}(n) \leq \left\lceil 10n/3 + \sqrt{2n/3} \right\rceil$ [4] (announced in a slightly weaker form in [3]). He asks whether $\mathfrak{v}(n) = 3n - 2$ for all $n \geq 4$.

In this paper we improve the upper bound to $\mathfrak{v}(n) \leq 16n/5 + O(1)$:

Theorem 1.1 *Let $A_1, \ldots, A_n$ be n equivalence relations with kernels $K_1, \ldots,$ K_n, respectively. Suppose $|K_i| \geq (3 + 1/5)n + c$ for each i, where c is a large enough constant.*[3] *Then $A_1, \ldots, A_n$ have a rainbow matching.*

We prove Theorem 1.1 by a modification of Grinblat's argument. The proof follows by induction on the number of equivalence relations n, showing that given a rainbow matching (of $n-1$ pairs) for the equivalence relations $A_2, \ldots, A_n$, it is possible to obtain a rainbow matching for $A_1, A_2, \ldots, A_n$.

The proof follows in several steps, where in each step, we observe that it is either possible to complete a full fledged rainbow matching, or to slightly

[3] It is enough to take $c = 5000$.

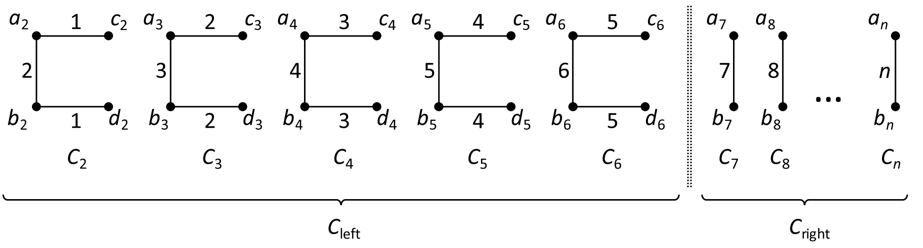

Fig. 2. Left-side and right-side components.

extend the previous formation at hand. The final formation allows us to complete the rainbow matching, concluding the proof.

2 Proof sketch of Theorem 1.1

Assume by induction on n that $A_2, \ldots, A_n$ have a rainbow matching $a_2 \sim_2 b_2$, $a_3 \sim_3 b_3$, …, $a_n \sim_n b_n$. Let $B = \{a_2, b_2, \ldots, a_n, b_n\}$.

Observation 2.1 *If there exist two distinct elements $a_1 \sim_1 b_1$ with $a_1, b_1 \in X \setminus B$ then we are immediately done.*

Lemma 2.2 *Let $t = \lfloor n/5 \rfloor$. Either we can easily complete a rainbow matching, or else we can find $t-1$ distinct indices in $\{2, \ldots, n\}$, which without loss of generality we assume to be $2, \ldots, t$, and we can find $2(t-1)$ pairwise distinct elements $c_2, d_2, \ldots, c_t, d_t \in X \setminus B$, such that $a_i \sim_{i-1} c_i$ and $b_i \sim_{i-1} d_i$ for all $2 \le i \le t$.*

Proof. Suppose by induction that we have already found $c_2, d_2, \ldots, c_i, d_i$.

Let $B' = B \cup \{c_2, d_2, \ldots, c_i, d_i\}$. Partition B' into "components" as follows: $C_2 = \{a_2, b_2, c_2, d_2\}$, …, $C_i = \{a_i, b_i, c_i, d_i\}$; $C_{i+1} = \{a_{i+1}, b_{i+1}\}$, …, $C_n = \{a_n, b_n\}$. Let $C_{\text{left}} = C_2 \cup \cdots \cup C_i$ and $C_{\text{right}} = C_{i+1} \cup \cdots \cup C_n$. See Figure 2.

Observation 2.3 *If there exist two distinct elements $x \sim_i y$, with $x, y \in K_i \setminus C_{\text{right}}$, then we are easily done* unless *one of x, y belongs to $\{a_j, c_j\}$ and the other one belongs to $\{b_j, d_j\}$ for the* same *index $2 \le j \le i$.*

Hence, let us charge each element of K_i to exactly one component, as follows:

Charging Scheme 1 Let $x \in K_i$. If $x \in B'$, then x is charged to the component it belongs to. Otherwise, by Observation 2.3, x must be i-equivalent to some $y \in C_{\text{right}}$; then we charge x to y's component. (If x can be charged to more than one right-side component, then we choose one of them arbitrarily.)

The total number of charges is equal to $|K_i|$, which is at least $(3+1/5)n+c$. By Observation 2.3 and the transitivity of $\sim_i$, no component can get more than four charges. Hence, if $i \leq n/5$, then there must be a component in C_{right} that received four charges. Without loss of generality it is C_{i+1}. Of the four elements charged to it, the two not belonging to it are the desired c_{i+1}, d_{i+1}.

This concludes the proof of Lemma 2.2. □

Define the set B', the components $C_2, \ldots, C_n$, and the sets C_{left} and C_{right} as above, with t in place of i. Hence, $C_{\text{left}} = C_2 \cup \cdots \cup C_t$ and $C_{\text{right}} = C_{t+1} \cup \cdots \cup C_n$.

We now use the following charging scheme for A_1 and A_t:

Charging Scheme 2 Consider an element $x \in K_1$. If $x \in B$, then we 1-charge x to the component it belongs to. Otherwise, by Observation 2.1, x must be 1-equivalent to some element $y \in B$; then we 1-charge x to the component that contains y.

Consider the elements of K_t. We t-charge every element $z \in (K_t \cap B)$ to the component it belongs to. If $c_i \sim_t d_i$ for some i, then we t-charge both elements to the component C_i that contains them (even if they are also t-equivalent to some element of C_{right}). For every $z \in K_t$ not covered by the above cases, by Observation 2.3 there must be a component C_i that contains an element $y \sim_t z$ (furthermore, either C_i is a right-side component, or else $z \in C_i$); we charge z to C_i.

Lemma 2.4 *In Charging Scheme 2, no component C_i can receive more than four 1-charges, or more than four t-charges.*

For each $2 \leq i \leq n$, let σ_i (resp. τ_i) be the number of 1-charges (resp. t-charges) that component C_i received; let S_i (resp. T_i) be set of elements *not* in $\{a_i, b_i\}$ that were 1-charged (resp. t-charged) to C_i; and let $U_i = S_i \cup T_i$.

Lemma 2.5 *Suppose that there exist five different left-side components that receive at least 7 charges each; namely, suppose there exist $C_{i_1}, \ldots, C_{i_5}$, with $2 \leq i_1 < \cdots < i_5 \leq t$, such that $\sigma_{i_k} + \tau_{i_k} \geq 7$ for each $1 \leq k \leq 5$. Then we can complete a rainbow matching.*

No component can receive more than 8 charges. Hence, by Lemma 2.5, there must be at least $n/5 + c - 4 \geq n/5$ right-side components that receive at least 7 charges each. Call such components "heavy", and let H be the set of their indices; namely, let

$$H = \{i \in \{t+1, \ldots, n\} : \sigma_i + \tau_i \geq 7\}.$$

Lemma 2.6 *Let C_i, C_j, $i, j \in H$, be two heavy components. Then we can find four distinct elements $v_1, v_2 \in C_i \cup C_j$, $w_1, w_2 \in U_i \cup U_j$, such that $v_1 \sim_1 w_1$ and $v_2 \sim_t w_2$.*

Furthermore, for any two fixed elements $q, r \in U_i \cup U_j$, it is always possible to do so guaranteeing that exactly one *of q, r belongs to $\{w_1, w_2\}$.*

For $i \in H$, let us call a left-side component *i-tainted* if it intersects T_i. For each i there are at most two i-tainted components.

Lemma 2.7 *Let C_i, $i \in H$ be a heavy component.*

(a) If there exist two distinct elements $x \sim_i y$, both outside $B \cup S_i$, then we are done.

(b) Let C_j be a left-side component that is not i-tainted. Then, if one of a_j, b_j is i-equivalent to an element outside $B' \cup T_i$, then we are done.

Let us i-charge the elements of $K_i \setminus U_i$ to components according to the following charging scheme (which is similar to Charging Scheme 1):

Charging Scheme 3 Let $i \in H$. Consider an element $x \in K_i \setminus U_i$. If $x \in B'$, then x is i-charged to the component it belongs to. Otherwise, if x is i-equivalent to an element of S_i or to a_j or b_j where component C_j is i-tainted, then x is not i-charged. Otherwise, x must be i-equivalent to an element $y \in C_{\text{right}}$; then we charge x to the component that contains y.

Lemma 2.8 *In Charging Scheme 3, no component receives more than four i-charges.*

We have $|K_i \setminus U_i| \geq |K_i| - 4$, and there are at most six elements of this set that are not charged. Hence, there are at least $n/5 + c - 10$ components that received at least four charges. Out of them, at least $c - 10$ are right-side components.

Let us apply this charging for all $i \in H$. By the pigeonhole principle, there must be a "lucky" right-side component $C_{j^\star}$ that receives four i-charges for all $i \in H'$, for some subset $H' \subseteq H$ of size $|H'| = (c - 10)/4$. For each index $k \in H'$, let W_k be the set of two elements not in C_{j^*} that were k-charged to C_{j^*}. Note that each W_k is disjoint from $B' \cup U_k$.

Let k_1, k_2 be a pair of distinct indices in H'. We would like to choose four distinct elements w, x, y, z, with $C_{j^\star} = \{w, x\}$ and $y, z \in W_{k_1} \cup W_{k_2}$, such that $w \sim_{k_1} y$ and $x \sim_{k_2} z$. If such a choice is not possible, then call the pair k_1, k_2 "conflicting".

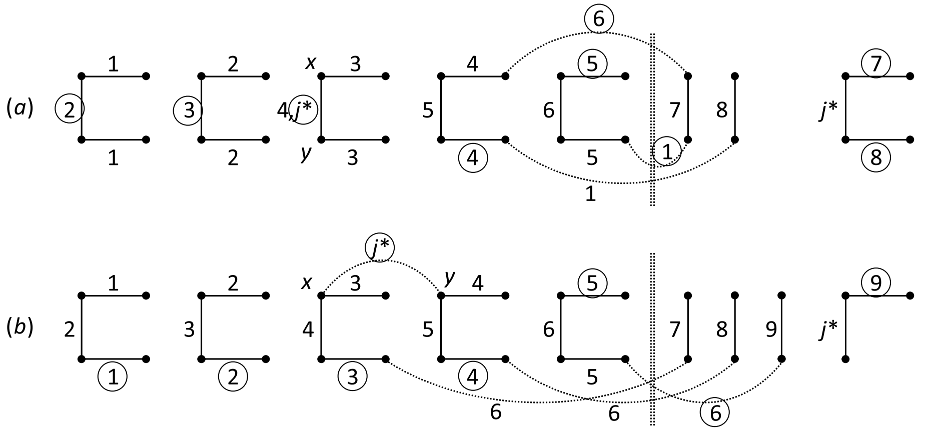

Fig. 3. Proof of Lemma 2.11.

Lemma 2.9 *There exists a subset of H' of size $|H'|/2$, such that no two indices in it are conflicting.*

Let $H'' \subset H'$ be such a subset, but only of size $|H''| = c/16$. Let $W = \bigcup_{k \in H''} W_k$.

Let k_1, k_2 be a pair of indices in H''. We already know that k_1, k_2 are not conflicting. We further would like to have $W_{k_1} \cap U_{k_2} = W_{k_2} \cap U_{k_1} = \emptyset$. If that is the case, call the pair k_1, k_2 "compatible".

Lemma 2.10 *Suppose the constant c is chosen large enough. Then most pairs of indices $k_1, k_2 \in H''$ are compatible; in particular, there exists such a compatible pair of indices.*

Lemma 2.11 *If there exist two elements $x \sim_{j^\star} y$, both outside*

$$Y = C_{\text{right}} \cup W \cup \Big(\bigcup_{k \in H''} U_k\Big),$$

then we are done.

Proof sketch See Figure 3. □

But a pair x, y as in Lemma 2.11 must exist, since otherwise, every element of $K_{j^\star}$ would either belong to Y or be $j^\star$-equivalent to a different element of Y. Observe that $|Y| \leq 2(4/5)n + c/8 + c/4$. That accounts for only $2|Y| \leq (3+1/5)n + 3c/4$ elements of $K_{j^\star}$, which is not enough. This concludes the proof of Theorem 1.1. □

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