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A stronger bound for the strong chromatic index

Extended Abstract

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Abstract

We prove $\chi'_s(G) \leq 1.93\Delta(G)^2$ for graphs of sufficiently large maximum degree where $\chi'_s(G)$ is the strong chromatic index of G. This improves an old bound of Molloy and Reed. As a by-product, we present a Talagrand-type inequality where it is allowed to exclude unlikely bad outcomes that would otherwise render the inequality unusable.

Keywords: Strong chromatic index, Maximum Degree, Edge coloring

1 Introduction

Edge colorings are well understood, for strong edge colorings this is much less the case. An edge coloring can be viewed as a partition of the edge set of a graph G into matchings; the smallest such number of partition classes is the *chromatic index* of G. If we consider the natural stronger notion of a partition into *induced* (or *strong*) matchings, we arrive at the *strong chromatic index* $\chi'_s(G)$ of G, the minimal number of induced matchings needed.

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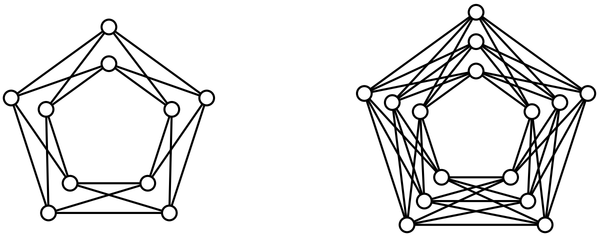

Fig. 1. Two blow-ups of the 5-cycle

The classic result of Vizing, and independently Gupta, constrains the chromatic index of a (simple) graph G to a narrow range: it is either equal to the trivial lower bound of the maximum degree $\Delta(G)$, or one more than that. The strong chromatic index, in contrast, can vary much more. The trivial lower bound and a straightforward greedy argument give a range of $\Delta(G) \leq \chi'_s(G) \leq 2\Delta(G)^2 - 2\Delta(G) + 1$ for all graphs G. Erdős and Nešetřil [7] conjectured a much stricter upper bound:

Conjecture 1.1 *$\chi'_s(G) \leq \frac{5}{4}\Delta(G)^2$ for all graphs G.*

If true, the conjecture would be optimal, because any blow-up of the 5-cycle as in Figure 1 attains equality. For odd maximum degree, Erdős and Nešetřil conjectured that $\chi'_s(G) \leq \frac{5}{4}\Delta(G)^2 - \frac{1}{2}\Delta(G) + 1$, which again would be tight.

In a breakthrough article of 1997, Molloy and Reed [18] demonstrated how probabilistic coloring methods could be used to beat the trivial greedy bound:

$$\chi'_s(G) \leq 1.998\Delta(G)^2$$

for graphs G with $\Delta(G)$ sufficiently large.

We improve this bound:

Theorem 1.2 *If G is a graph of sufficiently large maximum degree Δ, then*

$$\chi'_s(G) \leq 1.93\Delta^2.$$

A strong edge coloring of a graph G may be viewed as an ordinary vertex coloring in the square $L^2(G)$ of the linegraph of G. (The square of any graph is obtained by adding edges between any vertices of distance of most 2.) Working in $L^2(G)$ permitted Molloy and Reed to split the strong edge coloring problem into two weaker sub-problems. First, they showed that the neighborhood of any vertex in $L^2(G)$ is somewhat sparse. Second, based on a probabilistic coloring method, they proved a coloring result for graphs with sparse neighborhoods, that holds for general graphs, not only for squares of linegraphs.

We follow these same steps but make a marked improvement in each subproblem: our sparsity result is asymptotically best-possible; and our coloring lemma needs fewer colors.

As a tool for our coloring lemma we present in Section 3 a version of Talagrand's inequality that excludes exceptional outcomes. Talagrand's inequality is used to verify that random variables on product spaces are tightly concentrated around their expected value. It is particularly suited for random variables that only change little when a single coordinate is modified. This will not be the case in our application: in some very rare events a single change might result in a very large effect. To cope with this, we formulate a version of Talagrand's inequality in which such large effects of tiny probability can be ignored. We take some effort to make the application of the inequality as simple as possible as we have some hopes that it might be useful elsewhere.

A weakening of the strong edge coloring conjecture yields a statement on *strong* cliques, the cliques of the square of the linegraph.

Conjecture 1.3 *Any strong clique of a graph G has size at most $\omega(L^2(G)) \leq \frac{5}{4}\Delta^2(G)$.*

Note that the edge set of any blow-up of the 5-cycle is a strong clique, so that the conjecture would be tight. Not much is known about this seemingly easier conjecture. Chung, Gyárfás, Tuza and Trotter [4] showed that any graph G that is $2K_2$-free has at most $\frac{5}{4}\Delta^2(G)$ edges. In such a graph, the whole edge set forms a strong clique. Faudree, Schelp, Gyárfás and Tuza [8] found an upper bound of $(2-\epsilon)\Delta^2(G)$ for the size of any strong clique, for some small ϵ. Bipartite graphs are easier to handle in this respect: the same authors proved that the strong clique can never have size larger than $\Delta(G)^2$. Again, this is tight, as balanced complete bipartite graphs attain that bound.

We prove:

Theorem 1.4 *If G is a graph with maximum degree $\Delta \geq 400$, then its strong clique has size at most $\omega(L^2(G)) \leq 1.74\Delta^2$.*

Coming back to strong edge colorings, let us note that there are a number of results for special graph classes. The conjecture was verified for maximum degree 3 by Andersen [2], and independently by Horek, Qing and Trotter [10]. For $\Delta(G) = 4$, Cranston [5] achieves a bound of $\chi'_s(G) \leq 22$, which is off by 2. Finally, a number of works concern degenerated graphs, the earliest of which is by Faudree, Schelp, Gyárfás and Tuza [8] who established the bound $\chi'_s(G) \leq 4\Delta(G) + 4$ for planar graphs G. Kaiser and Kang [13] consider a generalization of strong edge colorings, where alike colored edges have to be even farther apart.

The use of probabilistic methods to color graphs is explored in depth in the book of Molloy and Reed [19], where also many more references can be found. We only mention additionally the article of Alon, Krivelevich and Sudakov [1] on coloring graphs with sparse neighborhoods. However, their result only implies something nontrivial if the neighborhoods are much sparser than we can expect in squares of linegraphs.

Strong edge colorings seem much more difficult than edge colorings. This is because already induced matchings are much harder to handle than ordinary matchings. While the size of a largest matching can be quite precisely be determined, it is even hard to obtain good bounds for induced matchings; see for instance [12,11].

All our graphs are simple and finite. We use standard graph-theoretic notation and concepts that can be found, for instance, in the book of Diestel [6].

2 Outline and proof of Theorem 1.2

A strong edge coloring of a graph G is nothing else than an ordinary vertex coloring in $L^2(G)$, the square of the linegraph of G. (The square of a graph is obtained by adding an edge between any two vertices of distance 2.) Molloy and Reed [18] use this simple observation to prove their bound on the strong chromatic index in two steps.

First, they show that neighborhoods in $L^2(G)$ cannot be too dense. To formulate this more precisely denote by N^{s}_e for any edge e of G the set of edges of distance at most 1 to e, which is equivalent to saying that N^{s}_e is the neighborhod of e in $L^2(G)$. We will often call N^{s}_e the *strong* neighborhood of e. Molloy and Reed show that for every edge e

$$N^{\mathsf{s}}_e \text{ induces in } L^2(G) \text{ at most } \left(1 - \tfrac{1}{36}\right)\binom{2\Delta^2}{2} \text{ edges,} \tag{1}$$

where Δ is the maximum degree of G.

In the second step, Molloy and Reed show that any graph with sparse neighborhoods, such as $L^2(G)$, can by colored with a probabilistic procedure.

Following this strategy, we also prove a sparsity result and a coloring lemma.

Lemma 2.1 *Let G be a graph of maximum degree $\Delta \geq 1$, and let e be an edge of G. Then the neighborhood N_e^s of e induces in $L^2(G)$ a graph of at most $\frac{3}{2}\Delta^4 + 5\Delta^3$ edges.*

Lemma 2.2 *Let $\gamma, \delta \in (0,1)$ be so that*

$$\gamma < \frac{\delta}{2(1-\gamma)} e^{-\frac{1}{1-\gamma}} - \frac{\delta^{3/2}}{6(1-\gamma)^2} e^{-\frac{7}{8(1-\gamma)}}. \tag{2}$$

Then, there is an integer R so that for all $r \geq R$ it follows $\chi(G) \leq (1-\gamma)r$ for every graph G with maximum degree at most r in which, for every vertex v, the neighborhoods $N(v)$ induce graphs of at most $(1-\delta)\binom{r}{2}$ edges.

Condition (2) might be slightly hard to parse. Therefore, let us remark that in a range of $\delta \in (0, 0.9]$

$$\gamma = 0.1827 \cdot \delta - 0.0778 \cdot \delta^{3/2}$$

satisfies the condition and is not too far away from the best-possible γ.

Our main theorem is a direct consequence of the two lemmas.

Proof. [Proof of Theorem 1.2] Let G be a graph with maximum degree Δ sufficiently large. With Lemma 2.1 we conclude that for every vertex v of $L^2(G)$ the neighborhood induces a graph of at most $(\frac{3}{4} + o(1))\binom{2\Delta^2}{2}$ edges. Therefore, we can apply Lemma 2 with $r = 2\Delta^2$, $\delta = 0.24$, and $\gamma = 0.035$. □

While Molloy and Reed developed this very neat proof technique, our contribution consists in improving its two constituent steps. In particular, in the sparsity lemma we improve Molloy and Reed's $\frac{1}{36}$ in (1) to roughly $\frac{1}{4}$. This is almost as good as possible: we construct graphs that asymptotically reach the upper density bound of Lemma 2.1. Our coloring lemma also yields a γ that is somewhat smaller than the corresponding γ of Molloy and Reed.

3 Talagrand's inequality and exceptional outcomes

To prove that a random variable on a product probability space is strongly concentrated around its expectation is a very common task, and consequently, a number of powerful tools have been developed for this, among them McDiarmid's, Azuma's or Talagrand's inequality [16,3,20]. All of these tools have in common that they require the random variable to be somewhat smooth.

Consider a family of probability spaces $((\Omega_i, \Sigma_i, \mathbb{P}_i))_{i=1}^n$, and let $(\Omega, \Sigma, \mathbb{P})$ be their product. One common smoothness assumption for a random variable

$X : \Omega \to \mathbb{R}$ is that *each coordinate has effect at most c*: whenever any two $\omega, \omega' \in \Omega$ differ in exactly one coordinate then $|X(\omega) - X(\omega')| \le c$.

In order to obtain strong concentration the effect c should not be to large compared to n. However, sometimes in very rare cases the effect might be large and otherwise fairly small, but the exceptional outcomes should not have a large impact on whether the random variable is strongly concentrated.

We present a version of Talagrand's inequality that excludes a very unlikely set Ω^* of *exceptional* outcomes that nevertheless spoils smoothness. Warnke [21] (but see also Kutin [15]) extended McDiarmid's inequality in a similar direction by considering a sort of *typical* effect. However, Warnke's inequality is still too weak for us. Grable [9] as well presents a concentration inequality that excludes exceptional outcomes, which would be suitable for our purposes, were it not for the fact that there is a serious error in its proof. McDiarmid [17], too, describes a Talagrand-type inequality that excludes an exceptional set. (Its main feature, though, is to allow permutations as coordinates.) The inequality, however, does not seem to be of much use to us either.

We mention that the powerful, but technical, method of Kim and Vu [14] can also handle large but unlikely effects.

To deal with exceptional outcomes, we need to define certificates. Given an exceptional set $\Omega^* \subseteq \Omega$ and $s, c > 0$, we say that X *has upward* (s,c)*-certificates* if for every $t > 0$ and for every $\omega \in \Omega \setminus \Omega^*$ there is an index set I of size at most s so that $X(\omega') > X(\omega) - t$ for any $\omega' \in \Omega \setminus \Omega^*$ for which the restrictions $\omega|_I$ and $\omega'|_I$ differ in less than t/c coordinates. Sometimes, it is easier to certify smaller values. To capture such situations we say that X *has downward* (s,c)*-certificates* if for every $t > 0$, and for every $\omega \in \Omega \setminus \Omega^*$ there is an index set I of size at most s so that $X(\omega') < X(\omega) + t$ for every $\omega' \in \Omega \setminus \Omega^*$ for which the restrictions $\omega|_I$ and $\omega'|_I$ differ in less than t/c coordinates.

Theorem 3.1 *Let* $((\Omega_i, \Sigma_i, \mathbb{P}_i))_{i=1}^n$ *be probability spaces,* $(\Omega, \Sigma, \mathbb{P})$ *be their product, and let* $\Omega^* \subset \Omega$ *be a set of exceptional outcomes. Let* $X : \Omega \to \mathbb{R}$ *be a random variable, let* $M = \max\{\sup |X|, 1\}$*, and let* $c \ge 1$*. Suppose* $\mathbb{P}[\Omega^*] \le M^{-2}$ *and* X *has upward* (s,c)*-certificates or downward* (s,c)*-certificates, then for* $t > 50c\sqrt{s}$

$$\mathbb{P}[|X - \mathbb{E}[X]| \ge t] \le 4e^{-\frac{t^2}{16c^2 s}} + 4\mathbb{P}[\Omega^*].$$

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