Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in DISCRETE MATHEMATICS

www.elsevier.com/locate/endm

An extension of the Bollobás-Riordan polynomial for vertex partitioned ribbon graphs: definition and universality

Thomas Krajewski [1,2]

Aix-Marseille Université
Marseilles, France

Iain Moffatt [1,3]

Royal Holloway University of London
Egham, United Kingdom

Adrian Tanasa [1,4]

Université Paris 13, Sorbonne Paris Cité
Villetaneuse, France

Abstract

In this paper we are interested in vertex partitioned ribbon graphs, which are a generalization of ribbon graphs that are studied in some theoretical physics models. We define a Hopf algebra of vertex partitioned ribbon graphs, then go on to describe how a natural generalization of the Bollobás-Riordan polynomial arises from this Hopf algebra. Using some appropriate Hopf algebraic characters we also prove the universality of our polynomial

Keywords: Graph polynomials, ribbon graphs, combinatorial Hopf algebras.

1 Hopf algebra of partitioned ribbon graphs

We study here vertex partitioned ribbon graphs. These are a natural generalization of ribbon graphs that appear in theoretical physics (in the *multi-trace matrix models*, see, for example, [5]). In addition to the physical applications, we will see that vertex partitioned ribbon graphs are useful in the study of deletion/contraction identities for the Bollobás-Riordan polynomial of [1]. In particular we will use them to describe a Hopf algebraic approach to the Bollobás-Riordan polynomial akin to that given for the Tutte polynomial of a matroid from [2].

A **ribbon graph** $\underline{G} = (V(\underline{G}), E(\underline{G}))$ is a surface with boundary, represented as the union of two sets of discs: a set $V(\underline{G})$ of vertices and a set of edges $E(\underline{G})$ such that vertices and edges intersect in disjoint line segments, each such line segment lies on the boundary of precisely one vertex and precisely one edge, and every edge contains exactly two such line segments (see Fig. 1, and [3] for additional background). Here we only consider orientable ribbon graphs (i.e., those that are orientable when considered as surfaces).

Definition 1.1 A **(vertex) partitioned ribbon graph** $G = (\underline{G}, \mathcal{P})$ is a ribbon graph $\underline{G}$ equipped with a partition $\mathcal{P}$ of its set of vertices $V(\underline{G})$. We refer to the blocks of the partition as **parts**. Its **shadow** $\overline{G}$ is the graph whose vertices are the parts of $\mathcal{P}$ with one edge between the parts for each edge of $\underline{G}$ between vertices in those blocks.

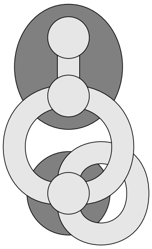
Note that the edges of the $\overline{G}$ are in 1-1 correspondence with the edges of $\underline{G}$. It is convenient to graphically represent the sets of the partition as balls on which the vertices of $\underline{G}$ are drawn as in Fig. 1.

For a partitioned ribbon graph G, its genus is $g(\underline{G}) = k(\underline{G}) - \frac{1}{2}\big(v(\underline{G}) - e(\underline{G}) + f(\underline{G})\big)$, rank is $r(\underline{G}) = v(\underline{G}) - k(\underline{G})$ and nullity is $n(\underline{G}) = e(\underline{G}) - r(\underline{G})$ where as usual, e, v, f and k represent the number of edges, vertices, boundary components and connected components. We will also consider the rank an nullity of its shadow graph $\overline{G}$,

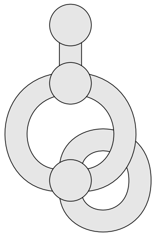
We say that G is a **join** of G' if $\underline{G}$ can be obtained from $\underline{G'}$ by identifying an arc on the boundary of a vertex in one component of $\underline{G'}$ with an arc on

[1] T. K. and A. T. acknowledge ESI Vienna. A. T. is partially supported by the PN 09370102 grant. T. K. and A. T. are partially supported by the ANR JCJC CombPhysMat2Tens grant.

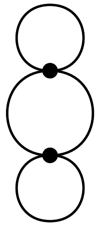
[2] Email: `thomas.krajewski@cpt.univ-mrs.fr`
[3] Email: `Iain.Moffatt@rhul.ac.uk`
[4] Email: `Adrian.Tanasa@lipn.univ-paris13.fr`



(a) A vertex partioned ribbon graph G.



(b) Its underlying ribbon graph $\underline{G}$.



(c) Its shadow $\overline{G}$.

Fig. 1. A vertex partitioned ribbon graph, its underlying ribbon graph and shadow

the boundary of a vertex in a different component (the identified vertices are merged into a single vertex of $\underline{G}$). Note that rank, nullity and genus take the same value on G and G'.

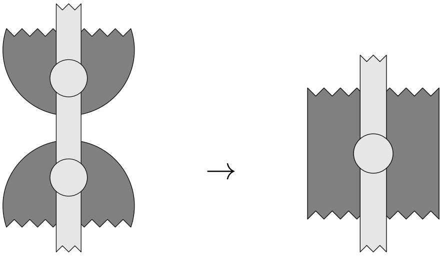
Let $\widetilde{\mathcal{H}}$ be the vector space over a zero characteristic ground field $\mathbb{K}$ that is freely generated by partitioned ribbon graphs. Defing a product as the disjoint union endows $\widetilde{\mathcal{H}}$ with an algebra structure. Moreover, one can define a coalgebra structure wiht the coproduct

$$\Delta(G) = \sum_{A \subseteq E(G)} G|_A \otimes G/A, \tag{1}$$

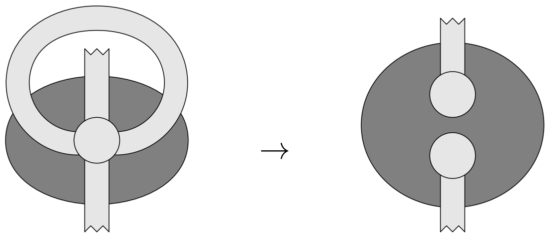
where the restriction $G|_A$ is the partitioned ribbon graph obtained by removing in $\underline{G}$ all the edges not in A and discarding isolated vertices. The vertex partition of $G|_A$ is induced from G. The contraction G/A is defined by, for each boundary component of $\underline{G}|_A \subseteq \underline{G}$, attaching a disc, which will form a vertex of G/e, by identifying its boundary component with the curve, then deleting the vertices and edges in the ribbon subgraph $\underline{G}|_A$. The vertex partition is obtained by, for each new vertex v of G/e, substituting v in each part of G for each vertex it has replaced in the construction of G/e then taking the union of all parts containing v. See Fig. 2.

The coproduct (1) leads to a Hopf algebra structure on partitioned ribbon graphs (for generalities on incidence Hopf algebras, see [6]). For this we identify graphs that differ by joins. We introduce the ideal $\mathcal{J} \subseteq \widetilde{\mathcal{H}}$ generated by the differences between partitioned ribbon graphs and their joins. This allows us to define a Hopf algebra structure on the space of equivalence classes of partitioned ribbon graphs differing by joins.

Proposition 1.2 *The coproduct* (1) *induces on* $\mathcal{H} = \widetilde{\mathcal{H}}/\mathcal{J}$ *the structure of a unital, connected, commutative Hopf algebra graded by the number of edges.*



(a) Contracting a non-loop edge.



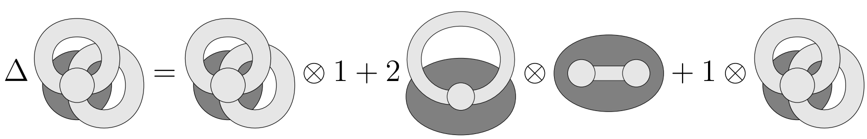
(b) Contracting a loop.

Fig. 2. Edge contraction.

Let us give a sketch of the proof. Since $\mathcal{J}$ is a Hopf ideal, one has $\Delta(\mathcal{J}) \subseteq \mathcal{H} \otimes \mathcal{J} + \mathcal{J} \otimes \mathcal{H}$. This implies that the coproduct Δ is well-defined on the quotient $\mathcal{H} = \widetilde{\mathcal{H}}/\mathcal{J}$. Note that one has the grading $\mathcal{H} = \oplus_n \mathcal{H}_n$, where $\mathcal{H}_n$ is spanned by the classes of graphs with n edges. This grading is compatible with the product and the coproduct. The unit 1 is the class of the edgeless partitioned ribbon graphs and the counit vanishes except on 1. Finally, the existence of an antipode follows from a standard inductive construction valid in commutative, graded and connected bigebras.

To simplify the notation, we identify graphs with their equivalence classes. Thus, the elements of $\mathcal{H}$ are identified with partitioned ribbon graphs and the coproduct (1) will acts on these equivalent classes. Example 1.3 below gives an illustration of this.

Example 1.3

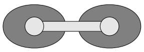


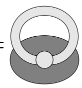
(2)

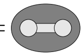
A **character** χ of $\mathcal{H}$ is a multiplicative linear map of it into $\mathbb{K}$, so $\chi(GG') = \chi(G)\chi(G')$. Since $\mathcal{H}$ is a commutative Hopf algebra, characters form a group for the convolution product defined by $\chi * \chi' = (\chi \otimes \chi') \circ \Delta$. An **infinitesimal**

character δ of $\mathcal{H}$ is a linear map to $\mathbb{K}$ such that $\delta(GG') = \delta(G)\epsilon(G') + \epsilon(G)\delta(G')$. Infinitesimal characters form a Lie algebra and the convolution exponential $\exp_*(\delta) = \sum_{n\geq 0} \frac{\delta^{*n}}{n!}$ (a finite sum when evaluated on any $G \in \mathcal{H}$) establishes a one to one correspondence between characters and infinitesimal characters.

There are three (orientable) partitioned ribbon graphs with one edge: the **coloop** (two vertices in different parts), the **loop** (one vertex) and the **semi-loop** (two vertices in one part). These give rise to three infinitesimal characters:

$\delta_c(G) = \{$ 1 if G =  ; 0 otherwise

$\delta_l(G) = \{$ 1 if G =  ; 0 otherwise

$\delta_s(G) = \{$ 1 if G =  ; 0 otherwise

(3)

Proposition 1.4 *The convolution exponential has the following interpretation.*

$$\exp_* \{a\delta_c + b\delta_l + c\delta_s\} (G) = a^{r(\overline{G})}\, b^{n(\underline{G})-g(\underline{G})}\, c^{n(\overline{G})-n(\underline{G})+g(\underline{G})}. \tag{4}$$

For any invertible elements $\lambda, \mu, \nu \in \mathbb{K}$, let us define the Hopf algebra automorphism of $\mathcal{H}$ by

$$\Psi_{\lambda,\mu,\nu}(G) = \lambda^{r(\overline{G})}\, \mu^{n(\underline{G})-g(\underline{G})} \nu^{n(\overline{G})-n(\underline{G})+g(\underline{G})}\, G. \tag{5}$$

One then has $\Psi_{\lambda,\mu,\nu}(\exp_* \{a\delta_{\mathrm{c}} + b\delta_{\mathrm{l}} + c\delta_{\mathrm{s}}\}) = \exp_* \{\lambda a\delta_{\mathrm{c}} + \mu b\delta_{\mathrm{l}} + \nu c\delta_{\mathrm{s}}\}$.

2 An extension of the Bollobás-Riordan polynomial

The Bollobás-Riordan polynomial [1] of a ribbon graph $\underline{G}$ is defined as

$$R_{\underline{G}}(x, y, z) = \sum_{A\subseteq E(\underline{G})} x^{r(\underline{G})-r(\underline{A})} y^{n(\underline{A})} z^{2g(\underline{A})}, \tag{6}$$

where the exponents refer to the spanning ribbon subgraphs of $\underline{G}$ defined by the edge set A. We now extend the Bollobás-Riordan polynomial to partitioned ribbon graphs.

Definition 2.1 The **extended Bollobás-Riordan polynomial** of a partitioned ribbon graph G is

$$R_G(x,y,z) = \sum_{A\subseteq E(G)} x^{r(\overline{G})-r(\overline{A})} y^{n(\overline{A})} z^{2\left(n(\overline{A})-n(\underline{A})+g(\underline{A})\right)}, \tag{7}$$

where $\overline{A}$ is the shadow graph of the spanning vertex partitioned ribbon graph defined by A.

This polynomial reduces to the ribbon graph Bollobás-Riordan polynomial if there is exactly one vertex in each part (i.e. one disc on each ball) since in this case $r(\underline{A}) = r(\overline{A})$ and $n(\underline{A}) = n(\overline{A})$. Moreover, when $z = 1$ we recover the Tutte polynomial (up to a shift of x and y by one) of the shadow graph. Alternatively, when $xyz^2 = 1$ there is no reference to $\overline{A}$ and the partition become irrelevant (up to a prefactor).

Theorem 2.2 *For any partitioned ribbon graph G,*

$$R_G(x,y,z) = \exp_*\left\{\delta_c + y\delta_l + yz^2\delta_s\right\} * \exp_*\left\{x\delta_c + \delta_l + \delta_s\right\}(G). \tag{8}$$

The proof of this theorem amounts to rewriting (7) using Proposition 1.4.

Corollary 2.3

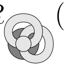
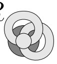
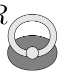
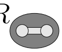
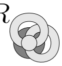
$$R_G(x,y,z) = \sum_{A\subseteq E(G)} R_{G|_A}(-1,y,z) R_{G/A}(x,-1,-1). \tag{9}$$

This follows from the convolution identity

$$\begin{aligned}&\exp_*\left\{\delta_\mathrm{c} + y\delta_\mathrm{l} + yz^2\delta_\mathrm{s}\right\} * \exp_*\left\{x\delta_\mathrm{c} + \delta_\mathrm{l} + \delta_\mathrm{s}\right\} = \\ &\exp_*\left\{\delta_\mathrm{c} + y\delta_\mathrm{l} + yz^2\delta_\mathrm{s}\right\} * \exp_*\left\{-\delta_\mathrm{c} + \delta_\mathrm{l} + \delta_\mathrm{s}\right\} * \exp_*\left\{\delta_\mathrm{c} - \delta_\mathrm{l} - \delta_\mathrm{s}\right\} * \exp_*\left\{x\delta_\mathrm{c} + \delta_\mathrm{l} + \delta_\mathrm{s}\right\},\end{aligned} \tag{10}$$

which we then rewrite as a sum over edge sets.

Example 2.4 The convolution identity for the extended Bollobás-Riordan polynomial of a genus one bouquet is shown below.

$\overbrace{R}$  $\overbrace{(x,y,z)}^{1+2y+y^2z^2} = \overbrace{R}$  $\overbrace{(-1,y,z)}^{1+2y+y^2z^2} + 2\,\overbrace{R}$  $\overbrace{(-1,y,z)}^{1+y}\ \overbrace{R}$  $\overbrace{(x,-1,-1)}^{1+(-1)\times(-1)^2=0} + \overbrace{R}$  $\overbrace{(x,-1,-1)}^{1-2+1=0}$ (11)

3 Universality of the polynomial R_G

Let $P_G(a,b,c,a',b',c')$ be a six variable polynomial defined on partitioned ribbon graph that obeys the following contraction/deletion rules (if G is edgeless, we define $P_G = 1$)

$$P_G(a,b,c,a',b',c') = \theta(e)P_{G/e}(a,b,c,a',b',c') + \theta'(e)P_{G\backslash e}(a,b,c,a',b',c'), \quad (12)$$

where

$$\theta(e) = \begin{cases} a \text{ if } e \text{ is neither a loop nor a semi-loop} \\ b \text{ if } e \text{ is a loop} \\ c \text{ if } e \text{ is a semi-loop} \end{cases} \qquad \theta'(e) = \begin{cases} a' \text{ if } e \text{ is a bridge (in } \overline{G}) \\ b' \text{ if } e \text{ is a planar edge} \\ c' \text{ if } e \text{ is a non-planar edge} \end{cases} \quad (13)$$

Here we call an edge e **planar** (resp. **non planar**) if its ends are attached to the same boundary component of $G\backslash e$ (resp. its ends are attached to two different boundary components of $G\backslash e$). Moreover, the edge e is a **bridge** if $\overline{G}\backslash e$ has one more connected component than $\overline{G}$. Note that there are seven instead of nine case to study since an edge cannot be at the same time a loop (or a semi-loop) and a bridge.

We now state our universality theorem (see [1,4] for the universality theorem for the ribbon graph Bollobás-Riordan polynomial):

Theorem 3.1 *Let $P_G(a,b,c,a',b',c')$ be a six variable partitioned ribbon graph polynomial which is multiplicative over disjoint unions and joins, and obeys the contraction/deletion relations (12) for any edge (normalized to unity for edgeless graphs). One then has:*

$$P_G(a,b,c,a',b',c') = a^{r(\overline{G})}\, b'^{n(G)-g(G)}\, c'^{n(\overline{G})-n(G)+g(G)}\, R_G\Big(\frac{a'}{a}, \frac{b}{b'}, \frac{c}{c'}\Big). \quad (14)$$

To establish this result, we define the Hopf algebra character χ by $\chi(G) = P_G(sa, sb, sc, sa', sb', sc')$. The contraction/deletion relations (12) then imply the following differential equation:

$$\frac{d\chi}{ds} = \big(a\delta_{\rm c} + b\delta_{\rm l} + c\delta_{\rm s}\big) * \chi + \chi * \big(a'\delta_{\rm c} + b'\delta_{\rm l} + c'\delta_{\rm s}\big). \quad (15)$$

Since the polynomial is normalized for edgeless graphs, $\chi = \epsilon$ (counit) for $s = 0$, the solution of the differential equation writes:

$$\chi = \exp_*\{sa\delta_{\rm c} + sb\delta_{\rm l} + sc\delta_{\rm s}\} * \exp_*\{sa'\delta_{\rm c} + sb'\delta_{\rm l} + sc'\delta_{\rm s}\}. \quad (16)$$

Acting then with the automorphism $\Psi_{1/a,1/b',1/c'}$ (assuming that none of these numbers vanish) as in (5) yields

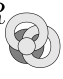
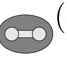
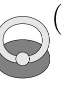
$$\Psi_{1/a,1/b',1/c'}(\chi) = \exp_*\left\{s\delta_{\mathrm{c}} + s\frac{b}{b'}\delta_{\mathrm{l}} + s\frac{c}{c'}\delta_{\mathrm{s}}\right\}*\exp_*\left\{s\frac{a'}{a}\delta_{\mathrm{c}} + s\delta_{\mathrm{l}} + s\delta_{\mathrm{s}}\right\}. \quad (17)$$

Evaluating both sides on a partitioned ribbon yields the announced result (14), since the action of the automorphism reduces to a multiplication by $a^{-r(\overline{G})}\, b'^{-n(G)+g(G)}\, c'^{-n(\overline{G})+n(G)-g(G)}$. If a, b' or c' vanish, the result still holds since the negative powers cancel.

Finally, let us notice that the extended Bollobás-Riordan polynomial also obeys contraction/deletion relations for any edge.

Example 3.2

$$\overbrace{R\;(x,y,z)}^{1+2y+y^2z^2} = \overbrace{yR\;(x,y,z)}^{y(1+yz^2)} + \overbrace{R\;(x,y,z)}^{1+y} \quad (18)$$

References

[1] B. Bollobás and O. Riordan, O, *A polynomial of graphs on surfaces*, Math. Ann. **323** (2002), 81–96.

[2] G. Duchamp, N. Hoang-Nghia T. Krajewski and A. Tanasa, *Recipe theorem for the Tutte polynomial for matroids, renormalization group-like approach*, Adv. in Appl. Math. **51** (2013), 345–358.

[3] J.A. Ellis-Monaghan and I. Moffatt, "Dualities, polynomials, and knots", Springer, New York, 2013.

[4] J.A. Ellis-Monaghan and I. Sarmiento,*A recipe theorem for the topological Tutte polynomial of Bollobás and Riordan*, European J. Combin. **32** (2011), 782–794.

[5] D. Malyshev, *Hopf algebra of ribbon graphs and renormalization*, J. High Energy Phys. **0205** (2002), 013.

[6] W. R. Schmitt, *Incidence Hopf algebras*, J. Pure Appl. Algebra **96** (1994), 299–330.