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Harmonious and achromatic colorings of fragmentable hypergraphs

Michał Dębski [1,2]

Faculty of Mathematics, Informatics and Mechanics
University of Warsaw, Warsaw, Poland

Zbigniew Lonc, Paweł Rzążewski [1,3]

Faculty of Mathematics and Information Science
Warsaw University of Technology, Warsaw, Poland

Abstract

A *harmonious* coloring of a k-uniform hypergraph H is a rainbow vertex coloring such that each k-set of colors appears on at most one edge. A rainbow coloring of H is *achromatic* if each k-set of colors appears on at least one edge. The *harmonious* (resp. *achromatic*) *number* of H, denoted by $h(H)$ (resp. $\psi(H)$) is the minimum (resp. maximum) possible number of colors in a harmonious (resp. achromatic) coloring of H. A class $\mathcal{H}$ of hypergraphs is *fragmentable* if for every $H \in \mathcal{H}$, H can be fragmented to components of a bounded size by removing a „small" fraction of vertices.

We show that for every fragmentable class $\mathcal{H}$ of bounded degree hypergraphs, for every $\epsilon > 0$ and for every hypergraph $H \in \mathcal{H}$ with $m \geq m_0(\mathcal{H}, \epsilon)$ edges we have $h(H) \leq (1+\epsilon)\sqrt[k]{k!m}$ and $\psi(H) \geq (1-\epsilon)\sqrt[k]{k!m}$.

As corollaries, we answer a question posed by Blackburn (concerning the maximum length of packing t-subset sequences of constant radius) and derive an asymptotically tight bound on the minimum number of colors in a vertex-distinguishing edge coloring of cubic planar graphs (which is a step towards confirming a conjecture of Burris and Schelp).

Keywords: harmonious coloring, achromatic coloring, fragmentable hypergraphs

1 Introduction

A coloring of vertices of a k-uniform hypergraph H is *rainbow* if all vertices of every edge receive different colors. We say that such a coloring *covers* a k-subset A of the set of colors if there is an edge in H whose vertices receive colors from A. A *harmonious coloring* of a k-uniform hypergraph H is a rainbow coloring (of vertices of H) that covers each k-subset of the set of colors at most once. The *harmonious number* of H, denoted by $h(H)$, is the least possible number of colors in such a coloring.

Similarly, an *achromatic coloring* of H is a rainbow coloring which covers every k-subset of the set of colors. The *achromatic number* of H, denoted by $\psi(H)$, is the largest possible number of colors in an achromatic coloring of H (or 0 if there is no achromatic coloring of H).

One can easily observe a kind of duality between achromatic and harmonious colorings. However, it is worth mentioning that although harmonious coloring of a k-uniform hypergraph always exists, there are k-uniform hypergraphs (with $k \geq 3$) that do not have an achromatic coloring. For example the hypergraph whose edges are all k-element subsets of an n-element set of vertices that contain a fixed vertex has no achromatic coloring, if $2 < k < n$.

Both these chromatic parameters have been well-studied for graphs – see the survey of Edwards [5] for a summary of results in this area. From our perspective, the most important achievements related to the topic of this paper are theorems of Edwards and Cairine [4] and Edwards [6], which state that if G is a sufficiently large graph with bounded degree, then both $\psi(G)$ and $h(G)$ stay within a factor of $(1 \pm \epsilon)$ from $\sqrt{2m}$, where m is the number of edges of G.

Let us observe that harmonious and achromatic colorings of hypergraphs are closely related to some other combinatorial problems. For example existence of a harmonious coloring of the hypergraph tK_q^k (where tK_q^k is the disjoint union of $t = \binom{n}{k}/\binom{q}{k}$ copies of the complete k-uniform hypergraph on q vertices) into n colors is equivalent to existence of a Steiner system $S(k, q, n)$ – a hard combinatorial problem recently solved for fixed k and q and sufficiently large n by Keevash [10].

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[2] Email: `michal.debski87@gmail.com`
[3] Email: `{zblonc, p.rzazewski}@mini.pw.edu.pl`

2 New results

A class of hypergraphs $\mathcal{H}$ is *fragmentable* if for every $\epsilon > 0$ there is an integer $c = c(\epsilon)$ such that if $H \in \mathcal{H}$ is a hypergraph with n non-isolated vertices then there is a set S of vertices with $|S| \leq \epsilon n$ such that each component of $H - S$ has at most c vertices. Intuitively, a hypergraph is fragmentable if it can be split to fragments of size bounded by a constant by removing a small portion of vertices. Fragmentable classes of graphs were introduced and first studied by Edwards and McDiarmid [7]. Our definition is a natural extension of this concept to hypergraphs.

We show asympotically tight bounds on achromatic and harmonious numbers of uniform fragmentable hypergraphs of bounded degree.

Theorem 2.1 *Let $\mathcal{H}$ be a fragmentable class of k-uniform hypergraphs of bounded maximum degree. For every $\epsilon > 0$ there exists some m_0 such that if $H \in \mathcal{H}$ has $m \geq m_0$ edges, then $h(H) \leq (1+\epsilon)\sqrt[k]{k!m}$.*

Theorem 2.2 *Let $\mathcal{H}$ be a fragmentable class of k-uniform hypergraphs of bounded maximum degree. For every $\epsilon > 0$ there exists some m_0 such that if $H \in \mathcal{H}$ has $m \geq m_0$ edges, then $\psi(H) \geq (1-\epsilon)\sqrt[k]{k!m}$.*

The complementary bounds can be proved easily: if we have a harmonious (resp. achromatic) coloring of a k-uniform hypergraph with m edges that uses x colors, then $\binom{x}{k}$ must be greater (resp. smaller) than m. Let $Q_{k,m}$ be the unique real root of the equation $\binom{x}{k} = \frac{x(x-1)\ldots(x-k+1)}{k!} = m$ which is not smaller than k. Then $h(H) \geq Q_{k,m}$ and $\psi(H) \leq Q_{k,m}$. Since k is a constant, $Q_{k,m} \sim \sqrt[k]{k!m}$.

It was shown by Edwards and McDiarmid [7] that many important classes of graphs are fragmentable. This includes the class of planar graphs (and more generally of graphs of bounded genus) and the class of rectangular lattices of bounded dimension. Actually, it follows from the work of Alon, Seymour and Thomas [1] that any class of graphs with a fixed excluded minor is fragmentable.

Several classes of hypergraphs obtained from fragmentable graphs by some natural constructions are fragmentable. As an example consider the class of dual hypergraphs of graphs with a bounded maximum degree. For a graph G, by the *dual hypergraph* $H(G)$ we mean the hypegraph whose vertices are edges of G and for every vertex v in G the set of edges incident with v is an edge in $H(G)$. We observe that if $\mathcal{G}$ is a fragmentable class of graphs of maximum degree at most d then the class of hypergraphs dual to the graphs in $\mathcal{G}$ is fragmentable too.

Clique hypergraphs (i.e. hypergraphs whose edges are maximal cliques of graphs) and *neighborhood hypergraphs* (i.e. hypergraphs whose edges are neighborhoods of vertices in graphs) obtained from graphs of a fragmentable class produce next examples of fragmentable classes of hypergraphs.

Face hypergraphs provide still another class of fragmentable hypergraphs. Given an embedding of a graph $G = (V, E)$ on a surface $\mathcal{S}$, its *face hypergraph* $F(G)$ is the hypergraph on V such that every face of G generates an edge in $F(G)$ consisting of the vertices incident to the face. This concept was introduced in [11] and its chromatic properties were studied by several authors (e.g. see [9]). As classes of graphs with a bounded genus are fragmentable, one can easily verify that for any fixed surface $\mathcal{S}$ of a finite genus and any class $\mathcal{G}$ of embeddings of graphs on $\mathcal{S}$, the class of face hypergraphs of graphs in $\mathcal{G}$ is fragmentable. Therefore our Theorems 2.1 and 2.2 imply some theorems on colorings of graphs in $\mathcal{G}$. For example, let $\mathcal{G}$ be the set of *plane triangulations*, i.e. embeddings of graphs on a plane such that each face is formed by a triangle. In this case Theorem 2.1 gives the following result.

Theorem 2.3 *For any plane triangulation with at most $\binom{n}{3}$ faces and maximum degree bounded by a constant there is a rainbow coloring of vertices with $n(1+o(1))$ colors such that the triples of colors assigned to vertices of each face are pairwise different.*

2.1 An application: vertex distinguishing edge coloring

Let G be a graph. We say that an edge coloring of G is *vertex-distinguishing* if for any two distinct vertices u, v the set of colors assigned to edges incident with u differs from the set of colors assigned to edges incident with v (that is, we can distinguish vertices by the sets of colors that they see). We say that an edge coloring of G is *strong* if it is proper and vertex-distiguishing. Clearly, every graph with no isolated vertices and edges has a strong coloring. The minimum possible number of colors in a strong coloring of G is denoted by $\chi'_s(G)$. A conjecture of Burris and Schelp [3] states that for every graph G that admits a strong coloring, the value of $\chi'_s(G)$ differs from its obvious lower bound by at most 1. To see what this lower bound is note that if G has n_d vertices of degree d, then $\binom{\chi'_s(G)}{d} \geq n_d$. Define $j(G)$ to be the minimum integer such that $\binom{j(G)}{d} \geq n_d$ for all d. Clearly $\chi'_s(G) \geq j(G)$ and the conjecture states that $\chi'_s(G) \leq j(G) + 1$ (see [3, Conjecture 1]). Despite a lot of work in this area, the problem is far from being solved.

Our Theorem 2.1 immediately implies some asymptotic version of the conjecture of Burris and Schelp for cubic planar graphs (which makes an improvement over known general upper bounds which are tight only up to a multiplicative constant; see [3, Theorem 11]). The same argument works for any other class of fragmentable, d-regular graphs (where d is a constant).

Theorem 2.4 *For every $\epsilon > 0$ there is some n_0 such that if G is a cubic planar* *graph on $n > n_0$ vertices, then $\chi'_s(G) \le (1+\epsilon)j(G)$.*

2.2 An application: packing t-subset sequences

Let A be a finite alphabet of order a and let t and r be positive integers such that $t \le r+1$. Consider a sequence $s = s_1 s_2 \ldots s_n$ over A. We say that s is a *packing t-subset sequence of radius r* if every t-subset of A appears at most once as a subset of $\{s_i, s_{i+1}, \ldots, s_{i+r}\}$ – or, more formally, if for every t-subset X of A there exists at most one increasing sequence $i_1, \ldots, i_t$ such that $i_t - i_1 \le r$ and $\{s_{i_1}, s_{i_2}, \ldots, s_{i_t}\} = X$. Let $F_{t,r}(a)$ denote the length of the longest such sequence.

Blackburn [2] asks for a good asymptotic lower bound on $F_{t,r}(a)$ (where t and r are fixed and the size of the alphabet goes to infinity). A quick application of our Theorem 2.1 gives an answer to this question.

Theorem 2.5 *Let r and $t \le r+1$ be fixed positive integers. For every $\epsilon > 0$ there is a_0 such that, for $a \ge a_0$,*

$$(1-\epsilon)\frac{\binom{a}{t}}{\binom{r}{t-1}} \le F_{t,r}(a) < \frac{\binom{a}{t}}{\binom{r}{t-1}} + r. \tag{1}$$

3 Overview of the used methods

Before more general results appeared, Edwards and McDiarmid [7] proved the upper bound of $(1+\epsilon)\sqrt{2m}$ on the harmonious number of fragmentable graphs of bounded degree (a corresponding lower bound on the achromatic number of those graphs follows as a corollary). The general idea of their proof is the following. Given a graph G, remove a "small portion" of its vertices to obtain a graph G' whose components are pairwise isomorphic and have order bounded by a constant (since G belongs to a fragmentable class of graphs, G' retains almost all edges of G). Next use Wilson's theorem [12] to find a harmonious coloring of G' – a partition of K_h into edge-disjoint copies of components of G' corresponds to a harmonious coloring of G' with h colors. Finally, color the remaining vertices of G using a small number of new colors.

This proof does not work for hypergraphs for two reasons. First of them is that there is no hypergraph version of Wilson's theorem. The second one is more subtle and lies in the final step of the reasoning presented above when we need to color ϵn vertices with new colors. A direct adaptation of the method used in [7] may give a coloring with $\Theta(m^{1-\frac{1}{k}})$ new colors which is too many, for $k > 2$, as to prove our theorem we are allowed to use at most $O(\sqrt[k]{m})$ colors.

We can overcome the first difficulty if we replace the use of Wilson's theorem with a theorem of Frankl and Rödl [8]. This idea was inspired by Blackburn [2], who used a similar approach to deal with a different, but related problem. The covering variant of the Frankl-Rödl theorem is sufficient to prove our Theorem 2.2.

However, we are not able to overcome the second obstacle using the Frankl-Rödl theorem as a black box. Fortunately, the proof of this theorem (which relies on so-called Rödl nibble method) can be modified to suit our needs.

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