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Properly colored and rainbow copies of graphs with few cherries [1]

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Abstract

Let G be an n-vertex graph that contains linearly many cherries (i.e., paths on 3 vertices), and let c be a coloring of the edges of the complete graph K_n such that at each vertex every color appears only constantly many times. In 1979, Shearer conjectured that such a coloring c must contain a properly colored copy of G. We prove this conjecture even for graphs G with $O(n^{4/3})$ cherries and show that this is up to a constant factor best possible. We also prove an analogue of this conjecture for colorings of $E(K_n)$ where for each color the total number of appearances is bounded, and then the aim is to find a rainbow copy of G.

Our proofs combine a framework of Lu and Székely for using lopsided Lovász local lemma in the space of random injections together with some other ideas.

Keywords: Ramsey-type problems, rainbow and properly colored subgraphs, spanning subgraphs, local lemma.

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1 Introduction

The canonical version of Ramsey theorem [8] for graphs implies that for every graph G, there exists an integer n such that any coloring of the edges of the complete graph K_n contains at least one of the following copies of G:

- *monochromatic* copy, i.e., a copy where all the edges have the same color,
- *rainbow* copy, which is a copy where no two edges have the same color, or
- *lexicographic* copy, in which case the vertices of the copy can be ordered in such a way that the color of any edge is purely determined by the smaller endpoint.

Note that by restricting the number of colors that the coloring of $E(K_n)$ can use, the copy must be monochromatic which proves the classical Ramsey theorem.

In this work, we consider the following two different types of restrictions, which are kind of dual to bounding the number of colors: we do not allow any color to, either locally or globally, appear too many times. More precisely, we say that a coloring c of $E(K_n)$ is *locally k-bounded* if for every vertex $v \in V(K_n)$, no color appears more than k-times on the $(n-1)$ edges incident to v. Analogously, we say that c is *globally k-bounded* if no color appears more than k-times on all the $\binom{n}{2}$ edges of K_n.

The copies of G we want to find in such colorings are either properly colored or rainbow. We define that a coloring c of $E(K_n)$ is *G-proper*, if there exists a copy of G in K_n for which c induces its proper edge-coloring. Similarly, we say that c is *G-rainbow* if there exists a copy of G in K_n such that no two edges of this copy have the same color in c.

1.1 Locally bounded colorings and properly colored subgraphs

A conjecture of Bollobás and Erdős [5] from 1976 states that every locally $(n/2)$-bounded coloring of $E(K_n)$ is C_n-proper, i.e., it contains a properly colored Hamilton cycle. In [5], they proved a weaker result – any locally cn-bounded coloring is C_n-proper, where the constant c is equal to $1/69$. Around the same time, Chen and Daykin [7] proved the same result for $c = 1/17$, and then in 1979, Shearer [17] improved the value of c to $1/7$. After another improvement due to Alon and Gutin [3], Lo [14] proved that cn-locally bounded coloring is C_n-proper for any $c < 1/2$ and n sufficiently large.

In [17], Shearer proposed the following generalization of the conjecture to an arbitrary graph G that do not contain too many *cherries*, i.e., paths on three vertices.

Conjecture 1.1 (Shearer [17]) *For every two integers s and k, there exists an integer n_0 such that the following is true. If $n \geq n_0$ and G is an n-vertex graph with at most sn cherries, then any locally k-bounded coloring of $E(K_n)$ is G-proper.*

Our main result is that we prove this conjecture. Actually we prove an even stronger statement, where we allow the graphs G to have up to $\Theta(n^{4/3})$ cherries.

Theorem 1.2 *If G is an n-vertex graph with at most r cherries, then any locally $\left(\frac{n}{560r^{3/4}}\right)$-bounded coloring c of $E(K_n)$ is G-proper.*

On the other hand, there are locally 3-bounded colorings c_n of $E(K_n)$ together with n-vertex trees T_n with $\Theta(n^{4/3})$ cherries so that c_n is not T_n-proper.

Another generalization of the conjecture of Bollobás and Erdős to a general graph G takes into account the maximum degree. Alon, Jiang, Miller and Pritikin [4] showed that if G is an n-vertex graph with maximum degree Δ and $k = O\left(\frac{\sqrt{n}}{\Delta^{27/2}}\right)$, then any locally k-bounded coloring c of $E(K_n)$ is G-proper. Their result was then improved by Böttcher, Kohayakawa and Procacci [6] who showed that k can be of order n/Δ^2.

Theorem 1.3 ([6]) *If G is an n-vertex graph with maximum degree Δ, then any locally $(n/22.4\Delta^2)$-bounded coloring c of $E(K_n)$ is G-proper.*

Our next contribution is that we show, up to a constant factor, tightness of this result for all values n and Δ. Even more, we can find graphs G with maximum degree Δ and locally $(3.9n/\Delta^2)$-bounded but not G-proper colorings, where the number of vertices of G does not depend on n at all.

Proposition 1.4 *For every prime power q and integer n, there exist an ℓ-vertex graph G with maximum degree Δ, where $\ell = q^2 + q + 1$ and $\Delta = q + 1$, and a locally $(3.9n/\Delta^2)$-bounded coloring c of $E(K_n)$ so that c is not G-proper.*

1.2 Globally bounded colorings and rainbow subgraphs

There is a rich literature studying rainbow copies of a fixed graph in globally bounded colorings of $E(K_n)$, see for example [2,10,12,13]. In this work, we will focus on finding rainbow spanning subgraphs.

Various authors have considered an analogue of Bollobás-Erdős conjecture, where the aim is to find a rainbow Hamilton cycle in a globally bounded coloring of $E(K_n)$. Specifically, Hahn and Thomassen [11] conjectured that there is a constant $c > 0$ such that any globally cn-bounded coloring of K_n is C_n-rainbow. Their conjecture was proven by Albert, Frieze, and Reed [1] with $c = 1/64$ (see also [16] for a correction of the originally claimed constant).

In 2008, Frieze and Krivelevich [9] showed that there is some absolute constant $c > 0$ so that any globally cn-bounded coloring actually contain copies of C_k for all $k \in \{3, \ldots, n\}$. In the same paper, they conjectured that there is also a constant $c > 0$ such that every globally cn-bounded coloring contains any spanning tree with bounded maximum degree. Using the same technique as for proving Theorem 1.3, Böttcher, Kohayakawa and Procacci [6] proved the conjecture of Frieze and Krivelevich not only for trees, but actually for all spanning subgraphs with bounded maximum degree.

Theorem 1.5 ([6]) *If G is an n-vertex graph with maximum degree Δ, then any globally $(n/51\Delta^2)$-bounded coloring c of $E(K_n)$ is G-rainbow.*

With a slight modification of the construction from Proposition 1.4, we can show that the dependency $k = O(n/\Delta^2)$ in Theorem 1.5 is again best possible.

Proposition 1.6 *For every two integers Δ and n such that Δ is even and $\left(\frac{\Delta}{2}+1\right)^2$ divides n, there exist an n-vertex graph G with maximum degree Δ and a globally $(16n/\Delta^2)$-bounded coloring c of $E(K_n)$ so that c is not G-rainbow.*

Finally, we have found natural to ask what can we say about globally bounded colorings and rainbow copies of graphs that does not have bounded maximum degree, but contains only few cherries. We were able to obtain the following theorem, which is the analogue of Conjecture 1.1 in this setting.

Theorem 1.7 *If G is an n-vertex graph with at most r cherries, then any globally $\left(\frac{n}{1512r^{3/4}}\right)$-bounded coloring c of $E(K_n)$ is G-rainbow.*

Since the locally 3-bounded coloring c of $E(K_n)$ which shows the tightness of Theorem 1.2 is also globally 9-bounded, we conclude that again the number of cherries cannot exceed $\Theta(n^{4/3})$.

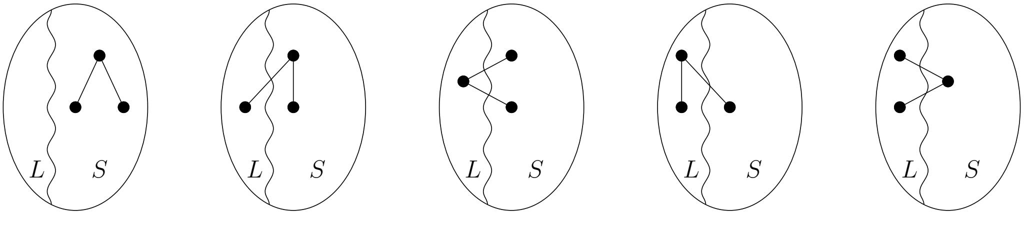
2 Sketch of the proof of Theorem 1.2

The main idea of Böttcher, Kohayakawa and Procacci [6] for proving Theorems 1.3 and 1.5 was to embed G randomly into K_n and then use local lemma through a framework of Lu and Székely [15]. In order to bound the negatively correlated dependencies for the lopsided version of local lemma, they needed the maximum degree of G to be of order $O(\sqrt{n})$.

One of the new ingredients for our theorem is to first perform some preprocessing that deals with the vertices of G that has degree larger than $\sqrt{n}$, and then we embed the rest randomly. Since the number of cherries of G is bounded by $O(n^{4/3})$, there can be only a small number of vertices of degree $\Omega(\sqrt{n})$. On the other hand, there is always a large enough subgraph $H \subseteq K_n$, so that the large-degree vertices of G fit there and have only a few monochromatic cherries in c with both leaves in $V(H)$.

Lemma 2.1 *For every n, k and r such that $r = O(n^{4/3})$ and $k = O(n/r^{3/4})$, the following is true. Every locally k-bounded coloring c of K_n contains a properly colored complete subgraph H of size $r^{1/4}$ such that for every two vertices $v_1, v_3 \in V(H)$, the set $\{v_2 \in V(K_n) : c(v_1v_2) = c(v_2v_3)\}$ has size $O(kr^{1/4})$.*

Let L be the first $r^{1/4}$ vertices of G according to their degrees and let $S := V(G) \setminus L$. It holds that maximum degree of $G[S]$ is $O(\sqrt{n})$. We map L to $V(H)$ arbitrarily and S to Q uniformly at random, where $Q := V(K_n) \setminus V(H)$. Since c restricted to $V(H)$ is a proper coloring of H, there are five types of monochromatic cherries we need to worry about:



In order to show that with positive probability none of the cherries of G is monochromatic, we use lopsided local lemma with a negative dependency graph D from the framework of Lu and Székely [15].

Before describing the graph D in details, let us introduce some additional notation. Let Ω be the probability space of random bijections π between S and Q. An event B is called *canonical* if there exist two sets $X \subseteq S$, $Y \subseteq Q$ and a bijection $\tau : X \to Y$ such that $B = \{\pi : \pi(a) = \tau(a) \text{ for all } a \in X\}$. For two sets $X \subseteq S$ and $Y \subseteq Q$ of the same size and a bijection $\tau : X \to Y$, we denote the corresponding canonical event by $\Omega(X, Y, \tau)$. We say that two canonical events $\Omega(X_1, Y_1, \tau_1)$ and $\Omega(X_2, Y_2, \tau_2)$ intersect if X_1 and X_2 intersect or Y_1 and Y_2 intersect.

Theorem 2.2 ([15]) *Let $\mathcal{B} = \{B_1, \ldots, B_N\}$ be a set of canonical events and D a graph with the vertex set $[N]$ where $ij \in E(D)$ if and only if the events*

B_i and B_j intersect. It holds that D is a negative dependency graph.

Using this D and analysing how the five types of cherries can intersect with each other, we conclude that with a positive probability the copy of G we constructed is properly edge-colored.

2.1 Locally 3-bounded coloring which is not T_n-proper

Finally, we describe the construction of c_n and T_n which shows that we cannot go over $O(n^{4/3})$ cherries. Let T_n be an n-vertex tree of radius two with exactly one vertex z of degree $\Theta(n^{2/3})$ that has all the neighbors of degree $\Theta(n^{1/3})$ and they have all their other neighbors of degree one. Note that T_n contains $\Theta(n^{4/3})$ cherries.

On the other hand, split arbitrarily the vertex-set $V(K_n)$ into n disjoint parts $P_1, \ldots, P_n$, each of size 3. The coloring c uses a palette of colors $[n] \cup \binom{[n]}{2}$ and two vertices $v_1 \in P_i$ and $v_2 \in P_j$ are colored with the color $\{i, j\}$. Note that if $i = j$, the edge $v_1 v_2$ is colored with the color $\{i\}$. It follows that c is locally 3-bounded.

Suppose there is a properly edge-colored copy of T_{3n} and let P_z be the part where the vertex z is mapped. If the two other vertices v_1 and v_2 in P_z are both neighbors of z in T_{3n}, then the vertices of P_z span a monochromatic cherry and the copy is not properly edge-colored. So one of the vertices, say v_1, must be a leaf of T_{3n}. However, then there is no part P_j where we can embed the common neighbor of z and v_1, a contradiction.

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