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10-tough chordal graphs are hamiltonian (Extended Abstract) [1]

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Abstract

Chen et al. proved that every 18-tough chordal graph has a hamiltonian cycle. Improving upon their bound, we show that every 10-tough chordal graph is hamiltonian. We use Aharoni and Haxell's hypergraph extension of Hall's Theorem as our main tool.

Keywords: chordal graph; toughness; hamiltonian cycle

1 Introduction

We study hamiltonian cycles and toughness in chordal graphs. Recall that following Chvátal [6], the *toughness* of a graph G is the minimum, taken over all separating sets S of vertices of G, of the ratio of $|S|$ to the number of components of $G \setminus S$. If G is complete, its toughness is defined to be ∞. We

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say that a graph is *t-tough* if its toughness is at least t. It is easy to observe that hamiltonian graphs are 1-tough. In the reverse direction, Chvátal [6] conjectured the following.

Conjecture 1.1 *There exists t_0 such that every t_0-tough graph is hamiltonian.*

Conjecture 1.1 is still open. The best available lower bound is due to Bauer, Broersma and Veldman [3] who constructed non-hamiltonian graphs with toughness arbitrarily close to $\frac{9}{4}$.

Partial results related to Chvátal's conjecture have been obtained in various restricted classes of graphs (see the survey [2] for details). A number of these results concern chordal graphs. For instance, it is known that every chordal planar graph of toughness more than 1 is hamiltonian [4], and so is every 1-tough interval graph [8] or every $\frac{3}{2}$-tough split graph [9]. All of these results are tight.

For chordal graphs in general, there are non-hamiltonian examples with toughness arbitrarily close to $\frac{7}{4}$, found in [3]. On the other hand, Chen et al. [5] showed that every 18-tough chordal graph is hamiltonian. In this paper, we improve the bound as follows:

Theorem 1.2 *Every 10-tough chordal graph is hamiltonian.*

Our main tool is a hypergraph extension of Hall's Theorem, due to Aharoni and Haxell [1].

2 Overspan graphs

Let G be a chordal graph. By a well-known theorem of Gavril [7], there exists a *tree representation* of G — that is, a tree T_0 and a family $\mathcal{F}$ of subtrees of T_0 such that G is isomorphic to the intersection graph of $\mathcal{F}$. We may assume that for each leaf of T_0, $\mathcal{F}$ contains a subtree consisting of the leaf as its only vertex. For each vertex v of G, let F_v denote the corresponding subtree in $\mathcal{F}$.

Let us fix the tree representation and modify T_0 to a tree T which we call the base tree for G. First, we fix an independent set I in G that is maximal with the property that for each $v \in I$, F_v is a path all of whose vertices have degree at most 2 in T_0. We choose I such that no member of I contains a subtree from $\mathcal{F}$ as a proper subgraph. Any path F_v with $v \in I$ is called an *I-path*; it is *trivial* if is consists of a single vertex. Let us colour all the edges of all I-paths red and colour the other edges of T_0 black.

Next, we suppress each degree 2 vertex of T_0 that is not an endvertex of any

I-path, one by one. The resulting tree T (the *base tree* for G) inherits a natural red-black colouring. Observe that any nontrivial I-path in T_0 corresponds to a red edge, the red edges form a matching and their endvertices are all of degree 2. Vertices of T_0 that exist also in T (that is, vertices of degree at least 3 and endvertices of I-paths) are called *substantial*.

We use T to construct a family of so-called *overspan graphs*, assigning one such graph A_e to each edge e of T. The vertex set of A_e is $V(G) \setminus I$. The graph may contain loops; to avoid ambiguity, we point out that we view a loop as an edge of a special type. Let r and s denote the endvertices of e, the edge set of A_e is defined as follows:

- there is a loop on a vertex u if F_u contains the vertices r and s in T_0,
- vertices u and v are joined by an edge if $r \in V(F_u)$ and $s \in V(F_v)$ (or vice versa), and uv is an edge of G.

Furthermore, for each black edge e of T we introduce an additional overspan graph A'_e which is a copy of A_e.

The reason for the name 'overspan graph' is that we view each edge of T as representing a gap that needs to be crossed by the sought hamiltonian cycle, and the edges of the associated overspan graph encode the ways to do so.

As the tree representation $(T_0, \mathcal{F})$ and the independent set I are fixed, let us use the notation $\mathcal{A}(G)$ for the family of overspan graphs for G.

For $\mathcal{B} \subseteq \mathcal{A}(G)$, we define a graph $G_{\mathcal{B}}$ on vertex set $V(G) \setminus I$, whose edge set is the union of the edge sets of all the graphs contained in $\mathcal{B}$; each edge is only included at most once in this union. For $\mathcal{B} = \mathcal{A}(G)$, we let the graph be denoted by $G_{\mathcal{A}}$.

We conclude this section by pointing out a connection between the family of overspan graphs and the hamiltonicity of G.

Lemma 2.1 *Let G be a chordal graph and $\mathcal{A}(G)$ the family of overspan graphs for G (with respect to a tree representation for G and a suitable independent set I). Assume that we can choose one edge from each graph in $\mathcal{A}(G)$ in such a way that the chosen edges form a matching (possibly including loops) in $G_{\mathcal{A}}$. Then G is hamiltonian.*

3 Hall's theorem for hypergraphs

In this section, we recall an extension of Hall's Theorem to hypergraphs due to Aharoni and Haxell [1]. We use this result as a tool to verify the condition in Lemma 2.1.

Let $\mathcal{A} = \{H_1, H_2, \dots, H_m\}$ be a family of hypergraphs. A *system of disjoint representatives* for $\mathcal{A}$ is a function $f : \mathcal{A} \to \bigcup_{i=1}^{m} H_i$ such that for all distinct $i, j \in \{1, \dots, m\}$, $f(H_i)$ is a hyperedge of H_i and $f(H_i) \cap f(H_j) = \emptyset$.

Recall that a *matching* in a hypergraph is a collection of pairwise disjoint hyperedges, and the *matching number* $\nu(H)$ of H is the size of a largest matching in H. A corollary of the main result of [1] is the following theorem.

Theorem 3.1 *Let $\mathcal{A}$ be a family of n-uniform hypergraphs. A sufficient condition for the existence of a system of disjoint representatives for $\mathcal{A}$ is that for every $\mathcal{B} \subseteq \mathcal{A}$, there exists a matching in $\bigcup \mathcal{B}$ of size greater than $n(|\mathcal{B}| - 1)$.*

The nontrivial direction of Hall's Theorem for graphs follows directly from the $n = 1$ case of Theorem 3.1. In our argument, we require the next case, $n = 2$, where the members of $\mathcal{A}$ are graphs. Indeed, we need to apply Theorem 3.1 to the family of overspan graphs $\mathcal{A}(G)$, which we regard as hypergraphs with hyperedges of size 1 (loops) and 2 (non-loops). Although the latter hypergraphs need not be uniform, it is not hard to reduce the problem to the uniform case.

Theorem 3.1 together with Lemma 2.1 imply the following sufficient condition for the hamiltonicity of G:

Lemma 3.2 *Let G be a chordal graph. If for every $\mathcal{B} \subseteq \mathcal{A}(G)$, the matching number of $G_{\mathcal{B}}$ is at least $2|\mathcal{B}| - 1$, then G is hamiltonian.*

4 Vertex covers of the overspan graphs and toughness

We sketch the proof of Theorem 1.2. Suppose that a 10-tough chordal graph G is not hamiltonian. By Lemma 3.2, $\nu(G_{\mathcal{B}}) \leq 2(|\mathcal{B}|-1)$ for some $\mathcal{B} \subseteq \mathcal{A}(G)$. We consider the vertex cover number of $G_{\mathcal{B}}$. Recall that this parameter (denoted by τ) is the minimum size of a set of vertices intersecting each edge of the given graph. By the classical theorem of König, $\nu(H) = \tau(H)$ for any bipartite graph H. Although $G_{\mathcal{B}}$ need not be bipartite, it can be shown to become bipartite after the removal of all vertices with loops; in fact, this implies that the same equality holds for $G_{\mathcal{B}}$:

Lemma 4.1 *The graph $G_{\mathcal{B}}$ satisfies $\nu(G_{\mathcal{B}}) = \tau(G_{\mathcal{B}})$.*

There is a crucial connection between the vertex cover numbers of unions of the overspan graphs and the toughness of G. Let C be a minimum vertex cover of $G_{\mathcal{B}}$; by Lemma 4.1, $|C| \leq 2(|\mathcal{B}| - 1)$. For technical reasons, we need to fix C and extend $\mathcal{B}$ to a maximal subfamily such that C is a vertex cover of $G_{\mathcal{B}}$. We will produce a separating set $S \subseteq V(G)$ demonstrating that G is not 10-tough; to find it, we augment C as follows.

Let B be the set of edges e of T such that $\mathcal{B}$ contains A_e. Let E' be the set of all red edges of B such that none of the adjacent (black) edges of T is contained in B. Any red edge e corresponds to an I-path, say F_{v_e}; let X' be the set of all vertices v_e of G such that $e \in E'$. We set $S = C \cup X'$ and show that it has the required properties.

Let E_* be the set of black edges contained in B. For $i \in \{0, 1, 2\}$, let $E_i \subseteq E_*$ consist of edges incident with exactly i vertices whose degree in T is at most 2. It is not hard to show that

$$|S| < 4\,|E_0| + 6\,|E_1| + 8\,|E_2| + 3\,|E'|\,. \tag{1}$$

Bounding the number of components of $G \setminus S$ is somewhat harder. Let us say that a vertex $v \in V(G)$ is *based* at a component L of $T \setminus (E_* \cup E')$ if L is the unique component containing a substantial vertex of F_v. An important observation is that each vertex of $G \setminus S$ is based at some component of $T \setminus (E_* \cup E')$, and adjacent vertices are based at the same component. We call a component K of $T \setminus (E_* \cup E')$ *real* if there is a vertex v of $G \setminus S$ based at K.

Using a discharging type argument, we bound the number of real components of $T \setminus (E_* \cup E')$ from below; the bound implies that

$$c(G \setminus S) > \frac{2}{5}\,|E_0| + \frac{3}{5}\,|E_1| + |E_2| + |E'|\,. \tag{2}$$

The details are technical and we omit them due to space restrictions. Comparing (1) and (2), we find that G is not 10-tough, a contradiction proving Theorem 1.2.

In conclusion, we remark that the bound of Theorem 1.2 is still far from the lower bound of 'almost' $\frac{7}{4}$ proved in [3], and there seems to be ample room for further improvements.

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