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Graph Cores via Universal Completability

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Abstract

A framework for a graph $G = (V, E)$, denoted $G(\mathbf{p})$, consists of an assignment of real vectors $\mathbf{p} = (p_1, p_2, \ldots, p_{|V|})$ to its vertices. A framework $G(\mathbf{p})$ is called universally completable if for any other framework $G(\mathbf{q})$ that satisfies $p_i^\mathsf{T} p_j = q_i^\mathsf{T} q_j$ for all $i = j$ and $(i,j) \in E$ there exists an isometry U such that $Uq_i = p_i$ for all $i \in V$. A graph is called a core if all its endomorphisms are automorphisms. In this work we identify a new sufficient condition for showing that a graph is a core in terms of the universal completability of an appropriate framework for the graph. To use this condition we develop a method for constructing universally completable frameworks based on the eigenvectors for the smallest eigenspace of the graph. This allows us to recover the known result that the Kneser graph $K_{n:r}$ and the q-Kneser graph $qK_{n:r}$ are cores for $n \geq 2r+1$. Our proof is simple and does not rely on the use of an Erdős-Ko-Rado type result as do existing proofs. Furthermore, we also show that a new family of graphs from the binary Hamming scheme are cores, that was not known before.

Keywords: universal rigidity, universal completability, graph cores

1 Introduction

Throughout we denote $[n] = \{1, \dots, n\}$. A matrix X is called positive semidefinite, denoted $X \succeq 0$, if all its eigenvalues are nonnegative. A *framework* for a graph $G = ([n], E)$ consists of an assignment of real vectors $\mathbf{p} = (p_1, p_2, \dots, p_n)$ to the vertices of G; We denote this by $G(\mathbf{p})$. Given a framework $G(\mathbf{p})$ define $\mathcal{C}_{\mathbf{p}}(G)$ to be the set of all n-by-n positive semidefinite matrices that satisfy $X_{ij} = p_i^{\mathsf{T}} p_j$ for all $i = j$ and $(i, j) \in E$. The *Gram matrix* of the vectors $p_1, \dots, p_n$, denoted $\mathrm{Gram}(p_1, \dots, p_n)$, is the n-by-n matrix whose ij entry is given by $p_i^{\mathsf{T}} p_j$ for all $i, j \in [n]$. A framework $G(\mathbf{p})$ is called *universally completable* if $\mathrm{Gram}(p_1, \dots, p_n)$ is the unique element in the set $\mathcal{C}_{\mathbf{p}}(G)$.

The notion of universally completability was introduced and studied in [4] due to its relevance to the psd matrix completion problem: Given a graph $G = (V = [n], E)$ and a vector $a \in \mathbb{R}^{E \cup V}$ indexed by the nodes and the edges of G, decide whether there exists a real symmetric $n \times n$ matrix X satisfying

$$X_{ij} = a_{ij} \text{ for all } \{i, j\} \in V \cup E, \text{ and } X \text{ is positive semidefinite.} \quad (1)$$

Universally completable frameworks provide a systematic method for constructing partial psd matrices that they admit a *unique* completion to a fully specified psd matrix. Such partial matrices have been a crucial ingredient for the study of two minor-monotone graph parameters considered in [5,2], defined in terms of ranks of psd matrix completions of G-partial matrices.

As noted in [4] the notion of universal completability is closely related to the well-studied notion of universal rigidity. Recall that a framework $G(\mathbf{p})$ is called *universally rigid* if for any other framework $G(\mathbf{q})$ satisfying $\|q_i - q_j\|_2 = \|p_i - p_j\|_2$ for all $(i, j) \in E$ then $\|p_i - p_j\|_2 = \|q_i - q_j\|_2$ for all $i, j \in [n]$. Geometrically this means that $G(\mathbf{p})$ can be obtained by $G(\mathbf{q})$ by a rigid motion of the Euclidean space.

To any framework $G(\mathbf{p})$ we associate the *extended framework*, denoted $(\nabla G, \hat{\mathbf{p}})$, where ∇G is the graph obtained from G by adding an apex node (labeled 0) and $\hat{p}_0 = 0$ and $\hat{p}_i = p_i$ for all $i \in [n]$. It was shown in [4] that $G(\mathbf{p})$ is universally completable if and only if $(\nabla G, \hat{\mathbf{p}})$ is universally rigid. Nevertheless, it is not known whether this equivalence extends to the more general setting of *tensegrity* frameworks.

One of the main results in [4] is a sufficient condition for showing that a framework $G(\mathbf{p})$ is universally completable. In the special case of bar frame-

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works (which we are concerned with in this paper), this turns out to be equivalent with Connelly's sufficient condition for universal rigidity [1].

Theorem 1.1 *Consider a graph $G = ([n], E)$ and a framework $G(\mathbf{p})$ in $\mathbb{R}^d$ such that $p_1, \dots, p_n$ span $\mathbb{R}^d$. Assume there exists $Z \in \mathcal{S}^n$ satisfying:*

(i) *Z is positive semidefinite;*

(ii) *$Z_{ij} = 0$ whenever $(i, j) \notin E$;*

(iii) *$\sum_{j \in [n]} Z_{ij} p_j = 0$ for all $i \in [n]$;*

(iv) *Z has corank d;*

(v) *For any $d \times d$ symmetric matrix R,*

$$p_i^\intercal R p_j = 0 \text{ for } i = j \text{ and } (i, j) \in E \Longrightarrow R = 0. \tag{2}$$

Then the framework $G(\mathbf{p})$ is universally completable.

A *graph homomorphism* is an adjacency preserving map from the vertex set of one graph to that of another. In other words, if G and H are graphs and φ is a function from $V(G)$ to $V(H)$, then φ is a homomorphism if $\varphi(u) \sim \varphi(v)$ in H whenever $u \sim v$ in G.

An important notion in the theory of graph homomorphisms is that of a *core*. A graph G is a core if every homomorphism from G to itself is an automorphism of G. A homomorphism from a graph to itself is referred to as an *endomorphism*, and an equivalent definition of a core is a graph which admits no endomorphism to a proper subgraph. For every graph G there exists a unique core $G^\bullet$ such that G admits homomorphisms both to and from $G^\bullet$, and $G^\bullet$ is called *the core of* G.

For a graph G, the *neighbourhood* of a vertex u, denoted $N_G(u)$, is the set of vertices in G adjacent to u. A homomorphism from a graph G to a graph H is said to be *locally injective* if it acts injectively on $N_G(u)$ for all $u \in V(G)$. Since homomorphisms preserve adjacency, we can equivalently say that a homomorphism φ is locally injective if and only if $\varphi(u) \neq \varphi(v)$ whenever u and v are at distance two.

For connected graphs, locally injective endomorphisms are always automorphisms:

Theorem 1.2 *[6] If G is connected then every locally injective endomorphism of G is an automorphism.*

From the above theorem we immediately get:

Corollary 1.3 *A connected graph G is a core if and only if all of its endomorphisms are locally injective.*

This work builds on the following theorem which establishes an unexpected relationship between universal completability and graph cores.

Theorem 1.4 *Consider a graph $G = ([n], E)$ and suppose there exists a universally completable framework $G(\mathbf{p})$ with the following properties: (i) $p_i \neq p_j$ whenever $d(i,j) = 2$ (ii) $\langle p_i, p_i \rangle$ is constant for all $i \in [n]$ and (iii) $\langle p_i, p_j \rangle$ is constant for all $(i,j) \in E$. Then G is a core.*

The proof of Theorem 1.4 is omitted due to space limitations.

Our main objective in this work is to apply the sufficient condition given in Theorem 1.4 to show that certain families of graphs are cores. To achieve this we need a systematic way to generate universally completable frameworks. These frameworks should additionally satisfy conditions $(i) - (iii)$ of Theorem 1.4 but this is an additionally difficulty we address later.

1.1 Least Eigenvalue Frameworks

We now give a method to construct universally completable frameworks that uses the eigenvectors that correspond to the minimum eigenvalue of the adjacency matrix of the graph.

Definition 1.5 Consider a graph G, let A be its adjacency matrix and suppose it has minimum eigenvalue τ of multiplicity m. Let $u_1, \ldots, u_m$ be an orthonormal basis of the τ-eigenspace and let P be the $n \times m$ matrix whose columns are the u_i's. For $i \in [n]$ let $p_i \in \mathbb{R}^m$, be the rows of P. We call $G(\mathbf{p})$ a *least eigenvalue framework* for G.

As we now show least eigenvalue frameworks are good candidates for being universally completable. This is essentially because the matrix $Z = A - \tau I$ satisfies assumptions $(i) - (iv)$ of Theorem 1.1.

Theorem 1.6 *Let $G(\mathbf{p})$ be a least eigenvalue framework for G. Then $G(\mathbf{p})$ is universally completable if and only if it satisfies: $p_i^\intercal R p_j = 0$ for all $i = j$ and $i \sim j$ implies that $R = 0$.*

Proof. We only prove sufficiency. Suppose that no such nonzero matrix R exists. Let A be the adjacency matrix of G and let τ be its least eigenvalue. We apply Theorem 1.1 with $Z = A - \tau I$. Clearly, $A - \tau I \succeq 0$ and so condition (i) holds. Trivially condition (ii) also holds. Condition (iii) can be rewritten as $ZP = 0$ where P is the matrix whose rows are the p_i. But by the definition of least eigenvalue framework the columns of P are τ-eigenvectors of A and therefore $(A-\tau I)P=0$. For condition (iv), note that the corank of $A-\tau I$ is equal to the dimension of the τ eigenspace of A, which is exactly the dimension the p_i live in and span. For $Z=A-\tau I$, condition (v) is exactly the implication in the theorem statement that we assumed was true. By Theorem 1.1 $G(\mathbf{p})$ is universally completable. □

1.2 1-Walk-Regular Graphs

Having determined a way to generate universally completable frameworks (cf. Theorem 1.6) our next goal is to identify graph classes with sufficient regularity so as to guarantee that their least eigenvalue frameworks satisfy conditions $(i)-(iii)$ of Theorem 1.4. In this section we focus on the class of *1-walk-regular* graphs: A graph with adjacency matrix A is called 1-walk-regular if there exist $a_k, b_k \in \mathbb{N}$ for all $k \in \mathbb{N}$ such that $A^k \circ I = a_k I$ and $A^k \circ A = b_k A$. Here we use $\circ$ to denote the Schur matrix product.

Note that any 1-walk-regular graph is a_2 regular. Also, any graph which is vertex- and edge-transitive is easily seen to be 1-walk-regular. Other classes of 1-walk-regular graphs include distance regular graphs and, more generally, graphs which are a single class in an association scheme. We now show that least eigenvalue frameworks of 1-walk-regular graphs satisfy assumptions (ii) and (iii) of Theorem 1.4.

Theorem 1.7 *Let $G=([n],E)$ be 1-walk-regular with minimum eigenvalue τ of multiplicity m. For any least eigenvalue framework $G(\mathbf{p})$ we have that $\langle p_i,p_i\rangle=\frac{m}{n}$ for $i\in[n]$ and $\langle p_i,p_j\rangle=\frac{\tau m}{na_2}$ for $(i,j)\in E$.*

Proof. Let E_τ be the orthogonal projector onto the τ-eigenspace of G. Clearly $E_\tau=\mathrm{Gram}(p_1,\ldots,p_n)$. Since the matrix E_τ is a polynomial in A, and since G is 1-walk-regular there exist numbers a and b such that $E_\tau \circ I=aI$ and $E_\tau \circ A=bA$. Thus $\langle p_i,p_i\rangle=a$ for all $i\in[n]$ and $\langle p_i,p_j\rangle=b$ for all $(i,j)\in E$. Since E_τ is a projector we have $na=\mathrm{Trace}(E_\tau)=\mathrm{rank}(E_\tau)=m$ and thus $a=m/n$. Similarly using $E_\tau\circ A=bA$ it follows that $b=\frac{\tau m}{na_2}$. □

Since $\tau<0$ it follows that $\frac{\tau m}{na_2}<0$. This will be important in the next section. Combining Theorem 1.6 with Theorem 1.7 we obtain a sufficient condition for showing that a 1-walk-regular graph is a core.

Theorem 1.8 *Let G be 1-walk-regular and $G(\mathbf{p})$ a least eigenvalue framework for G. Assume that $G(\mathbf{p})$ satisfies: (i) $p_i\neq p_j$ whenever $d(i,j)=2$ and (ii) $p_i^\intercal R p_j=0$ for all $i=j$ and $i\sim j$ implies $R=0$. Then G is core.*

2 Applying the Sufficient Condition

2.1 Kneser and q-Kneser graphs

The Kneser graph $K_{n:r}$ has as vertices the r-subsets of $[n]$; Two vertices are adjacent if the corresponding sets are disjoint. The q-Kneser graph $qK_{n:r}$ has the r-dimensional subspaces of the finite vector space $\mathbb{F}_q^n$ as its vertices, and two subspaces are adjacent if they are *skew*, i.e. their intersection is the zero subspace. As a first application of Theorem 1.8 we show that Kneser and q-Kneser graphs are cores.

Theorem 2.1 *For $n \geq 2r+1$, both $K_{n:r}$ and $qK_{n:r}$ are cores.*

Even though this result is known, our proof method is interesting as it does not rely on the use of an Erdős-Ko-Rado type result, as do existing proofs [3].

Kneser and q-Kneser graphs are easily seen to be 1-walk-regular since they are both edge and vertex transitive. By Theorem 1.8 it suffices to exhibit a least eigenvalue framework that satisfies assumptions (i) and (ii). We only sketch the proof for q-Kneser graphs, the proof for Kneser graphs being similar.

Let P be a matrix with rows indexed by the r-dimensional subspaces of $\mathbb{F}_q^n$ (i.e. the vertices of $qK_{n:r}$) and columns by the lines (1-dimensional subspaces) of $\mathbb{F}_q^n$ defined by $P_{S,\ell} = \alpha$ if $\ell \subseteq S$ and β if $\ell \cap S = \{0\}$.

Furthermore suppose that α and β are chosen such that the row sum is zero. The precise values of α and β are not important but one suitable choice is $\alpha = [r]_q - [n]_q$ and $\beta = [r]_q$, where $[k]_q = \frac{q^k-1}{q-1} = \sum_{i=0}^{k-1} q^i$ is the number of lines contained in a k-dimensional subspace of $\mathbb{F}_q^n$.

It is known that the columns of P span the least eigenspace of $qK_{n:r}$. To any vertex of $qK_{n:r}$ we associate the S-row of P; we denote this by $p_S \in \mathbb{R}^{[n]_q}$. Notice that for $S \neq T$ we have that $p_S \neq p_T$. Furthermore, notice that the vectors p_S do not span $\mathbb{R}^{[n]_q}$ since they are all orthogonal to the all ones vector $\mathbf{1}$. In fact the vectors p_S actually span $\{\mathbf{1}\}^\perp$.

Note that to apply Theorem 1.1 it is sufficient for the vectors $p_1, \ldots, p_n$ to span a space of dimension d, not necessarily lie in $\mathbb{R}^d$. The theorem still holds by the same proof, however we must change R from a $d \times d$ symmetric matrix to a symmetric linear map from span($\mathbf{p}$) to itself. Thus it remains to show:

Theorem 2.2 *Let $R : \{\mathbf{1}\}^\perp \to \{\mathbf{1}\}^\perp$ be a linear map such that*

$$p_S^\intercal R p_T = 0 \text{ for all } S, T \in V(qK_{n:r}) \text{ such that } S \sim T. \tag{3}$$

Then $R = 0$.

The proof of Theorem 2.2 is omitted due to space limitations.

2.2 Hamming graphs

The graphs we will consider in this section are Cayley graphs for $\mathbb{Z}_2^n$, and more specifically graphs from the binary Hamming scheme. In particular, we mainly focus on the Cayley graph for $\mathbb{Z}_2^n$ whose connection set contains all elements of weight $n-k$ for some $k \leq n$. This graph is bipartite if $n-k$ is odd, and so we will implicitly assume that $n-k$ is even throughout this section, unless otherwise noted. Also, if $k \geq 1$ and $n-k$ is even, then the graph is not bipartite and has two isomorphic components corresponding to the even and odd Hamming weight elements of $\mathbb{Z}_2^n$. We will refer to the component consisting of the even weight vertices as $H_{n,k}$. These vertices form a subgroup isomorphic to $\mathbb{Z}_2^{n-1}$ and this is therefore still a Cayley graph. We will show that $H_{n,k}$ is a core if $k \in \{1,2,3\}$, $n \geq 3k$ and $n-k$ is even. To the best of our knowledge this class of graphs were not known to be cores before.

Note that $H_{n,k}$ is arc transitive and therefore 1-walk-regular; so Theorem 1.8 applies here. Unlike the example of the q-Kneser graphs in which we started with a least eigenvalue framework and then proved that it satisfied (2), here we will begin with a framework that satisfies (2) and then prove that it is a least eigenvalue framework. The framework $\mathbf{p} = (p_x : x \in V(H_{n,k})) \subseteq \mathbb{R}^n$ we will use is the following: To vertex x of $H_{n,k}$ we assign p_x defined entrywise as $(p_x)_i = (-1)^{x_i}$ for all $i \in [n]$. Note that these vectors are not normalized, but this will not affect our arguments and we can simply divide by $\sqrt{n}$ whenever necessary. It is not hard to see that, after normalization, this vector assignment has constant inner product $\frac{2k-n}{n}$ on the edges of $H_{n,k}$. Moreover, this framework is injective and one can show that $\operatorname{span}(p_x : x \in V(H_{n,k})) = \mathbb{R}^n$. It remains to show the following:

Lemma 2.3 *Let p_x for $x \in V(H_{n,k})$ be defined as above, and let R be an $n \times n$ matrix such that $p_x^\intercal R p_y = 0$ whenever $x = y$ or $x \sim y$. Then $R = 0$. Moreover, for $k \in \{1,2,3\}$, $n \geq 3k$, and $n-k$ even the framework $\mathbf{p} = (p_x : x \in V(H_{n,k}))$ is a least eigenvalue framework for $H_{n,k}$.*

As a Corollary of Lemma 2.3 we get that:

Theorem 2.4 *For $k \in \{1,2,3\}$, $n \geq 3k$, and $n-k$ even the graph $H_{n,k}$ is a core.*

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