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Winning fast in fair biased Maker-Breaker games

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Abstract

We study the $(a : a)$ Maker-Breaker games played on the edge set of the complete graph on n vertices. In the following four games – perfect matching game, Hamiltonicity game, star factor game and path factor game, our goal is to determine the least number of moves which Maker needs in order to win these games. Moreover, for all games except for the star factor game, we show how Red can win in the strong version of these games.

Keywords: Maker-Breaker games, biased games, fast strategy, strong games.

1 Introduction

Let a and b be two positive integers, let X be a finite set and $\mathcal{F} \subseteq 2^X$ be a family of the subsets of X. In the $(a : b)$ *positional game* $(X, \mathcal{F})$, two players take turns in claiming a, respectively b, previously unclaimed elements of X, with one of them going first. The set X is referred to as the *board* of the game, while the elements of $\mathcal{F}$ are referred to as the *winning sets*. When there is no risk of confusion on which board the game is played, we will just use $\mathcal{F}$ to denote the game. The integers a and b are referred to as *biases* of the players. When $a = b$, the game is said to be *fair*. If $a = b = 1$, the game is called *unbiased*. Otherwise, the game is called *biased*. If a player has a strategy to win against any strategy of the other player, this strategy is called *a winning strategy*.

In the $(a : b)$ *Maker-Breaker* positional game $(X, \mathcal{F})$, the two players are called *Maker* and *Breaker*. Maker wins the game $\mathcal{F}$ at the moment she claims all the elements of some $F \in \mathcal{F}$. If Maker did not win by the time all the elements of X are claimed by some player, then Breaker wins the game $\mathcal{F}$. In order to show that Maker wins the game as both first and second player, we will assume in this paper that Breaker starts the game (as being the first player can only be an advantage in Maker-Breaker games).

It is very natural to play Maker-Breaker games on the edge set of a given graph G. Here, we focus on the $(a : b)$ games played on the edge set of the complete graph on n vertices, K_n, where n is a sufficiently large integer. So, in this case the board is $X = E(K_n)$.

For example: in the *connectivity game*, $\mathcal{T}_n$, the winning sets are all spanning trees of K_n; in the *perfect matching game*, $\mathcal{M}_n$, the winning sets are all independent edge sets of size $\lfloor n/2 \rfloor$ (note that in case n is odd, this matching covers all but one vertex in K_n); in the *Hamiltonicity game*, $\mathcal{H}_n$, the winning sets are all Hamilton cycles of K_n; in the k-vertex-connectivity game, $\mathcal{C}_n^k$, for $k \in \mathbb{N}$, the winning sets are all k-vertex-connected graphs on n vertices.

It is not very difficult to see that Maker wins all aforementioned unbiased games. Therefore, we can ask the following question: *How quickly can Maker*

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win the game? Parameter $\tau_{\mathcal{F}}(a : b)$ denotes the shortest *duration* of the $(a : b)$ Maker-Breaker game $\mathcal{F}$, i.e. the least number of moves t such that Maker wins the $(a : b)$ game $\mathcal{F}$ within t moves. For completeness, we say that $\tau_{\mathcal{F}}(a : b) = \infty$ if Breaker wins the game $\mathcal{F}$.

It was shown in [8] that, for $n \geq 4$, $\tau_{\mathcal{T}_n}(1 : 1) = n - 1$, which is optimal. In [5] it was proved that $\tau_{\mathcal{M}_n}(1 : 1) = n/2 + 1$, when n is even and $\tau_{\mathcal{M}_n}(1 : 1) = \lceil n/2 \rceil$, when n is odd and also that $\tau_{\mathcal{H}_n}(1 : 1) \leq n + 2$ and $\tau_{\mathcal{C}_n^k}(1 : 1) = kn/2 + o(n)$. Hefetz and Stich in [7] showed that $\tau_{\mathcal{H}_n}(1 : 1) = n+1$, and Ferber and Hefetz [3] recently showed that $\tau_{\mathcal{C}_n^k}(1 : 1) = \lfloor kn/2 \rfloor + 1$.

We also look at another type of positional games. In the *strong* positional game $(X, \mathcal{F})$, the two players are called *Red* and *Blue*, and Red starts the game. The winner of the game is the *first* player who claims all the elements of one $F \in \mathcal{F}$. If none of the players manage to do that before all the elements of X are claimed, the game ends in a *draw*.

By *strategy stealing argument* (see [2]), Blue cannot have a winning strategy in the strong game. So, in every strong game, either Red wins, or Blue has a drawing strategy. For the games where the draw is impossible, we know that Red wins. Unfortunately, the existence of Red's strategy tells us nothing about how Red should play in order to win. Finding explicit winning strategies for Red can be very difficult. The results in [3,4] show that fast winning strategies for Maker in certain games can be used in order to describe the winning strategies for Red in the strong version of these games.

Coming back to Maker-Breaker games, we are particularly interested in the $(a : a)$ Maker-Breaker games on $E(K_n)$, for constant $a \geq 1$. Although these games are studied less than unbiased and the $(1 : b)$ games, they are also significant. Just a slight change in bias from $a = 1$ to $a = 2$ can completely change the outcome (and thus the course of the play) of some games (see [2]). One example is the *diameter-2* game (where the winning sets are all graphs with diameter at most 2). It was proved in [1] that Breaker wins the $(1 : 1)$ *diameter-2* game, but Maker wins the $(2 : 2)$ *diameter-2* game.

Not so much is known about fast winning strategies in the fair $(a : a)$ Maker-Breaker games, where a can be greater than 1. From the results in [6,8], we obtain that in the connectivity game $\tau_{\mathcal{T}_n}(a : a) = \lceil (n-1)/a \rceil$ for all relevant values of a, which is optimal, as by Maker's strategy no cycles are created. Our research is concentrated on fast winning strategies in four $(a : a)$ Maker-Breaker games, for $a \in \mathbb{N}$.

2 Fast Maker's strategies

Firstly, we take a look at the $(a : a)$ perfect matching game, $\mathcal{M}_n$. The case $a = 1$ is already proved in [5], and we claim the following theorem for all $a \geq 2$.

Theorem 2.1 *Let $a \in \mathbb{N}$. Then for every large enough n the following is true for the $(a : a)$ Maker-Breaker perfect matching game:*

$$\tau_{\mathcal{M}_n}(a : a) = \begin{cases} \frac{n}{2a} + 1 & , \text{ if } a = 1 \text{ and } n \text{ is even,} \\ \left\lceil \frac{n}{2a} \right\rceil - 1 & , \text{ if } 2a \mid n - 1 \\ \left\lceil \frac{n}{2a} \right\rceil & , \text{ otherwise.} \end{cases}$$

Secondly, we analyse the $(a : a)$ Maker-Breaker Hamiltonicity game, $\mathcal{H}_n$, and obtain the following result for $a \geq 2$. The case $a = 1$ is proved in [7].

Theorem 2.2 *Let $a \in \mathbb{N}$. Then for every large enough n the following is true for the $(a : a)$ Maker-Breaker Hamiltonicity game:*

$$\tau_{\mathcal{H}_n}(a : a) = \begin{cases} \frac{n}{a} + 1 & , \text{ if } a = 1 \text{ or } (a = 2 \text{ and } n \text{ is even}), \\ \left\lceil \frac{n}{a} \right\rceil & , \text{ otherwise.} \end{cases}$$

We study two more $(a : a)$ Maker-Breaker games whose winning sets are spanning graphs. More precisely, we are interested in factoring the graph K_n with stars and paths. For fixed $k \geq 2$, let P_k denote a path with k vertices, and let S_k denote a star with $k - 1$ leaves, $K_{1,k-1}$. Now, for all large enough n, such that $k \mid n$, we are interested in finding the winning strategies in the $(a : a)$ P_k-factor game, denoted by $\mathcal{P}_{k,n}$, and in the $(a : a)$ S_k-factor game, denoted by $\mathcal{S}_{k,n}$, where the winning sets are all path factors and star factors of K_n, respectively, on k vertices. We obtain the following.

Theorem 2.3 *Let $a \in \mathbb{N}$ and $k \in \mathbb{N}$. Then for every large enough n, such that $k \mid n$, the following is true for the $(a : a)$ Maker-Breaker P_k-factor game:*

$$\tau_{\mathcal{P}_{k,n}}(a : a) = \left\lceil \frac{(k-1)n}{ka} \right\rceil .$$

Theorem 2.4 *Let $a \geq 1$ and $k \geq 3$ be integers. Then for every large enough n, such that $k \mid n$, the following is true for the $(a:a)$ Maker-Breaker S_k-factor game:*

$$\tau_{\mathcal{S}_{k,n}}(a:a) \leq \begin{cases} \left\lceil \frac{(k-1)n}{ka} \right\rceil & , \textit{ if } a \nmid \frac{(k-1)n}{k}, \\ \frac{(k-1)n}{ka} + 1 & , \textit{ otherwise.} \end{cases}$$

3 Strong games

As we already mentioned, fast Maker's strategies in some games can be used to obtain the strategies for Red in the corresponding strong games.

If Maker can win *perfectly fast* in the game $\mathcal{F}$, i.e. if the number of moves, t, she needs to win is equal to the cardinality of the smallest winning set $F \in \mathcal{F}$, that immediately implies Red's win in the strong game. Indeed, as Red starts the game, Blue has no chance to fully claim any winning set in less than t moves. So, Red can play according to the strategy of Maker, without worrying about Blue's moves, by which Red will claim a winning set in t moves, thus winning the game.

From Theorem 2.1, we can see that Maker can win perfectly fast in the $(a:a)$ perfect matching game in all cases, but in case $a=1$. Therefore, we immediately see that for $a \neq 1$, Red has a winning strategy for the corresponding strong game. For $a=1$ the proof that Red wins the strong game appears in [3].

Similarly to the perfect matching game, from Theorem 2.2 we can immediately see that Red has a winning strategy for the strong $(a:a)$ Hamiltonicity game in all but two cases – the case $a=1$ and the case $a=2$ and n is even. The case $a=1$ appears in [3], and for the remaining case, the following theorem proves that Red can win the strong $(2:2)$ Hamiltonicity game.

Theorem 3.1 *For every large enough even n the following is true: Red has a strategy for the $(2:2)$ Hamiltonicity game to win within $\frac{n}{2}+1$ rounds.*

Every P_k-factor of K_n, for given $k \in \mathbb{N}$ such that $k \mid n$, has to have $n(k-1)/k$ edges. Therefore, from Theorem 2.3, we obtain that Maker can win perfectly fast in the $(a:a)$ game $\mathcal{P}_{k,n}$ and Red can use the winning strategy of Maker in this game to win in the corresponding strong game.

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