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On a Ramsey-type problem of Erdős and Pach

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Abstract

Erdős and Pach (1983) asked if there is some constant $C > 0$ such that for any graph G on $Ck \ln k$ vertices either G or its complement $\overline{G}$ has an induced subgraph on k vertices with minimum degree at least $\frac{1}{2}(k-1)$. They showed that the above statement holds with Ck^2 in place of $Ck \ln k$ but that it does not hold with $Ck \ln k / \ln \ln k$. We show that it holds with $Ck \ln^2 k$, answering their question up to a $\ln k$ factor.

Keywords: Ramsey theory, quasi-Ramsey numbers, graph discrepancy, probabilistic method

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1 Introduction

Recall that the (diagonal, two-colour) Ramsey number is defined to be the smallest integer $R(k)$ for which any graph on $R(k)$ vertices is guaranteed to contain a homogeneous set of order k — that is, a set of k vertices corresponding to either a complete or independent subgraph. The search for better bounds on $R(k)$, particularly asymptotic bounds as $k \to \infty$, is a challenging topic that has long played a central role in combinatorial mathematics (see [3,5]).

We are interested in a degree-based generalisation of $R(k)$ where, rather than seeking a clique or coclique of order k, we seek instead an induced subgraph of order (at least) k with high minimum degree (clique-like graphs) or low maximum degree (coclique-like graphs). Erdős and Pach [1] introduced this class of problems in 1983, and called them *quasi-Ramsey problems.* They for instance showed that gradually relaxing the degree requirement reveals a spectrum of Ramsey-type problems along which there is a sharp change in asymptotics at a certain point. Naturally, this point of change corresponds to a degree requirement of half the order of the subgraph sought.

The topic lay essentially dormant for decades, but we recently revisited it together with Pach [4]. In particular, we refined our understanding of the threshold for mainly what we referred to in [4] as the *variable quasi-Ramsey numbers* (which corresponds to the parenthetical 'at least' above). For any $\nu > 0$ and positive integer k, we showed that any graph G or its complement contains as an induced subgraph some graph on $\ell \geq k$ vertices with minimum degree at least $\frac{1}{2}(\ell - 1) + \nu$ provided that G has at least $k^{\Omega(\nu^2)}$ vertices, and this is in a sense sharp [4, Theorem 3].

In the present work we instead focus on the harder version of this problem, the determination of what we have called the *fixed quasi-Ramsey numbers* (where 'exactly' takes place of the parenthetical 'at least' above). Using a result on graph discrepancy, Erdős and Pach proved that there is a constant $C > 0$ such that for any graph G on at least Ck^2 vertices either G or its complement $\overline{G}$ has an induced subgraph on k vertices with minimum degree at least $\frac{1}{2}(k - 1)$. With an unusual random graph construction, they also showed that the previous statement does not hold with $C'k \ln k / \ln \ln k$ in place of Ck^2 for some constant $C' > 0$. They asked if it holds instead with $Ck \ln k$. Our main contribution here is to affirm this, up to a logarithmic factor, by showing the following.

Theorem 1.1 *There exists a constant $C > 0$ such that for k large enough and any graph G on $Ck \ln^2 k$ vertices, either G or its complement $\overline{G}$ has an* *induced subgraph on k vertices with minimum degree at least $\frac{1}{2}(k-1)$.*

Our proof of Theorem 1.1 has a number of different ingredients, including the use of graph discrepancy, a probabilistic thinning result, and a greedy algorithm that was inspired by similar procedures for max-cut and min-bisection.

To abide by page limits, we have had to omit one part of the proof. The missing details are available at `http://arxiv.org/abs/1411.4459`.

2 An auxiliary result via discrepancy

Our first step in proving Theorem 1.1 will be to apply the following result. This is a bound on a variable quasi-Ramsey number, which is similar to Theorem 3(a) in [4]. The idea of the proof of this auxiliary result is inspired by the sketch argument for Theorem 2 in [1], in spite of the error contained in that sketch (cf. [4]).

Theorem 2.1 *For any constant $\nu \geq 0$, there exists a constant $C = C(\nu) > 1$ such that for any graph G on $Ck \ln k$ vertices, G or its complement $\overline{G}$ has an induced subgraph on $\ell \geq k$ vertices with minimum degree at least $\frac{1}{2}(\ell-1) + \nu\sqrt{\ell-1}$.*

We use a result on graph discrepancy to prove Theorem 2.1. Given a graph $G = (V, E)$, the *discrepancy* of a set $X \subseteq V$ is defined as

$$D(X) := e(X) - \frac{1}{2}\binom{|X|}{2},$$

where $e(X)$ denotes the number of edges in the subgraph $G[X]$ induced by X. We use the following result of Erdős and Spencer [2, Ch. 7].

Lemma 2.2 (Theorem 7.1 of [2]) *Provided that n is large enough, then if $t \in \{1, \ldots, n\}$, any graph $G = (V, E)$ of order n satisfies*

$$\max_{S \subseteq V, |S| \leq t} |D(S)| \geq \frac{t^{3/2}}{10^3}\sqrt{\ln(5n/t)}.$$

Proof of Theorem 2.1 (Sketch). Let $G = (V, E)$ be any graph on at least $N = \lceil Ck \ln k \rceil$ vertices for a sufficiently large choice of C. For any $X \subseteq V$ and $\nu > 0$, we define the following skew form of discrepancy:

$$D_\nu(X) := \left| e(X) - \frac{1}{2}\binom{|X|}{2} \right| - \nu |X|^{3/2}.$$

We now construct a sequence $(H_0, H_1, \ldots, H_t)$ of graphs as follows. Let H_0 be G or $\overline{G}$. At step $i+1$, we form H_{i+1} from $H_i = (V_i, E_i)$ by letting $X_i \subseteq V_i$ attain the maximum skew discrepancy D_ν and setting $V_{i+1} := V_i \setminus X_i$ and $H_{i+1} := H[V_{i+1}]$. We stop after step $t+1$ if $|V_{t+1}| < \frac{1}{2}N$. Let $I^+ \subseteq \{1, \ldots, t\}$ be the set of indices i for which $D(X_i) > 0$. By symmetry, we may assume

$$\sum_{i \in I^+} |X_i| \geq \frac{1}{4}N. \tag{1}$$

Claim 2.3 *For any $i \in I^+$ and $x \in X_i$, $\deg_{H_i}(x) \geq \frac{1}{2}(|X_i|-1)+\nu(|X_i|-1)^{1/2}$.*

Proof. Write $|X_i| = n_i$. We are trivially done if $n_i = 1$, so assume $n_i \geq 2$. Suppose $x \in X_i$ has strictly smaller degree than claimed and set $X_i' := X_i \setminus \{x\}$. Then, since $i \in I^+$,

$$\begin{aligned} D_\nu(X_i') &\geq e(X_i') - \frac{1}{2}\binom{n_i - 1}{2} - \nu(n_i - 1)^{3/2} \\ &> e(X_i) - \frac{1}{2}\binom{n_i}{2} - \nu\sqrt{n_i - 1} - \nu(n_i - 1)^{3/2}. \end{aligned}$$

Note that $n_i^{3/2} > n_i^{1/2} + (n_i - 1)^{3/2}$, which by the above implies $D_\nu(X_i') > D_\nu(X_i)$, contradicting the maximality of $D_\nu(X_i)$. □

Claim 2.3 implies that we may assume for each $i \in I^+$ that $|X_i| \leq k-1$, or else we are done. Writing $I^+ = \{i_1, \ldots, i_m\}$, we can show the following.

Claim 2.4 *For any $\ell \in \{1, \ldots, m-3\}$, $D(X_{i_{\ell+3}}) \leq \frac{5}{6}D(X_{i_\ell})$.*

This slightly technical claim is in fact the engine of this proof, and contains the essential application of Lemma 2.2; however, to abide by space restrictions, we have chosen to omit its justification.

Claim 2.4 implies that $(5/6)^{(m-1)/3}D(X_{i_1}) \geq D(X_{i_m}) \geq 1$ (assuming for simplicity $m \equiv 1 \pmod 3$), which then implies

$$m - 1 \leq \frac{3\ln(D(X_{i_1}))}{\ln(6/5)} \leq \frac{6}{\ln(6/5)}\ln(k-1).$$

By (1), we deduce that at least one of the m sets X_i with $i \in I^+$ satisfies

$$|X_i| \geq \frac{N\ln(6/5)}{25\ln k}.$$

This last quantity is at least k by a choice of C sufficiently large, contradicting our assumption that $|X_i| \leq k-1$ for each $i \in I^+$. This completes the argument. □

3 Proof of Theorem 1.1

The proof of the theorem entails running an algorithm, one of whose stopping criteria is fulfilment of the hypothesis (for the right parameters) of the following thinning result from [4]. When that happens, we can immediately apply the result to obtain a k-subset of vertices of the desired type. We remark that the following has a short, probabilistic proof.

Lemma 3.1 ([4]) *For any $0 < c < 1$ and $\varepsilon > 0$, let k be large enough that $\exp(\frac{1}{2}\varepsilon^2(k-1)) > k$. If H is a graph of order $\ell \geq k$ such that $\delta(H) \geq c\ell$, then there exists $S \subseteq V(H)$ of order k such that $\delta(H[S]) \geq (c-\varepsilon)(k-1)$.*

By 'right parameters' above, we mean those in the following specific form of Lemma 3.1.

Corollary 3.2 *Let H be a graph of order $\ell \geq k$. If $\ell < 2k$ and $\delta(H) \geq \frac{1}{2}(\ell-1)+8\sqrt{(k-1)\ln k}+8$, then there exists $S \subseteq V(H)$ of order k such that $\delta(H[S]) \geq \frac{1}{2}(k-1)+\sqrt{(k-1)\ln k}$.*

As a subroutine, we make use of the following algorithm which is inspired by the greedy algorithm for max-cut or min-bisection.

Lemma 3.3 *Let $G = (V,E)$ be a graph of order n with $\delta(G) \geq \frac{1}{2}(n-1)+t$ for some number t. Let $\alpha \in [0,1]$ and let $a, b \in \mathbb{N}$ such that $a+b=n$. Then either there exists $A \subseteq V$ of size a such that $\delta(G[A]) \geq \frac{1}{2}a-1+\alpha t$, or there exists $B \subseteq V$ of size b such that $\delta(G[B]) \geq \frac{1}{2}b-1+(1-\alpha)t$.*

Proof. Take any $A \subseteq V$ of size a and let $B := V \setminus A$. If there exists $x \in A$ with $\deg_A(x) < \frac{1}{2}a-1+\alpha t$ and $y \in B$ with $\deg_B(y) < \frac{1}{2}b-1+(1-\alpha)t$, then move x to B and y to A, i.e. *swap* x and y. Note that when there is no such pair of vertices x, y we are done. We just need to prove that, if we keep iterating, then this procedure must stop at some point.

Consider the number of edges in $G[A]$ before and after we swap x and y. The number of edges in $G[A]$ increases by at least

$$\deg_A(y) - \deg_A(x) - 1 \geq \delta(G) - \deg_B(y) - \deg_A(x) - 1 \geq 1/2,$$

(where we subtracted 1 in case x and y are adjacent). This shows that we cannot continue to swap pairs indefinitely. □

At last we are ready to prove the main result. In fact, we prove something stronger.

Theorem 3.4 *There exist constants $D, D' > 0$ such that for k large enough and any graph G on $Dk \ln^2 k$ vertices, G or its complement $\overline{G}$ has an induced subgraph on k vertices with minimum degree at least $\frac{1}{2}(k-1) + D'\sqrt{k-1}/\ln k$.*

Proof. Set $m := 400k \ln k$, $D := 800C$, where $C = C(2)$ is defined according to Theorem 2.1, and $D' := 1/(4\sqrt{D})$. By Theorem 2.1, since $Cm \ln m \leq 800Ck \ln^2 k \leq Dk \ln^2 k \leq |V(G)|$, we find G or $\overline{G}$ has an induced subgraph H on $\ell \geq m$ vertices with $\delta(H) \geq \frac{1}{2}(\ell-1) + 2\sqrt{\ell-1}$.

Let $x = \ell \bmod k$ (so $x \in \{0, \ldots, k-1\}$). We can now apply Lemma 3.3 to H with $a = k+x$, $b = \ell - k - x$, $t = 2\sqrt{\ell-1}$ and $\alpha = 1/2$. Suppose this gives us a subset $A \subseteq V(H)$ of size a such that $\delta(H[A]) \geq \frac{1}{2}a - 1 + \sqrt{\ell-1}$. Then $k \leq a < 2k$ and, by our choice of m, we have that $\sqrt{\ell-1} - 1/2 \geq \sqrt{m-1} - 1/2 \geq 8\sqrt{(k-1)\ln k} + 8$, and so Corollary 3.2 yields a subset $A' \subseteq A$ of size k such that $\delta(H[A']) \geq \frac{1}{2}(k-1) + \sqrt{(k-1)\ln k} \geq \frac{1}{2}(k-1) + D'\sqrt{k-1}/\ln k$, as required. In case Lemma 3.3 does not produce such a set A, it gives instead a subset B of size $b = \ell - k - x \equiv 0 \pmod{k}$ (so $b \geq m - 2k \geq 398k \ln k$) such that $\delta(H[B]) \geq \frac{1}{2}(b-1) - \frac{1}{2} + \sqrt{\ell-1}$. We iteratively apply Lemma 3.3 to $H[B]$ in a binary search to find a desired induced subgraph as follows.

Set $G_0 = H[B]$. Let $\ell_0 := |V(G_0)| = b$ (so that $398k \ln k \leq \ell_0 \leq Dk \ln^2 k$ and $\ell_0 \equiv 0 \pmod{k}$) and set $t_0 := \sqrt{\ell-1} - \frac{1}{2} \geq \sqrt{\ell_0 - 1} - \frac{1}{2} = \Omega(\sqrt{\ell_0})$ (so that $\delta(G_0) \geq \frac{1}{2}(\ell_0 - 1) + t_0$). Suppose that G_i is given, where G_i has ℓ_i vertices with $\ell_i \equiv 0 \pmod{k}$ and $\delta(G_i) \geq \frac{1}{2}(\ell_i - 1) + t_i$ for some number t_i. Set $a_i = \lfloor \ell_i/2k \rfloor k$ and $b_i = \lceil \ell_i/2k \rceil k$ so that $a_i + b_i = \ell_i$ and $a_i \equiv b_i \equiv 0 \pmod{k}$. Apply Lemma 3.3 with $G = G_i$, $a = a_i$, $b = b_i$, $t = t_i$, and $\alpha = \frac{1}{2}$. Then we either obtain a set of vertices A_i of size a_i such that $\delta(G_i[A_i]) \geq \frac{1}{2}a_i - 1 + \frac{1}{2}t_i$, in which case we set $G_{i+1} = G_i[A_i] = H[A_i]$, or we obtain a set of vertices B_i of size b_i such that $\delta(G_i[B_i]) \geq \frac{1}{2}b_i - 1 + \frac{1}{2}t_i$, in which case we set $G_{i+1} = G_i[B_i] = H[B_i]$. Now set $\ell_{i+1} = |V(G_{i+1})|$ and note that $\ell_{i+1} \equiv 0 \pmod{k}$ and $\delta(G_{i+1}) \geq \frac{1}{2}(\ell_{i+1} - 1) + t_{i+1}$, where $t_{i+1} = \frac{1}{2}(t_i - 1)$. Note also that $\ell_{i+1}/k \leq \lceil \ell_i/2k \rceil$.

In this way we obtain subgraphs $G_0, G_1, \ldots$ of $G_0 = H[B]$ and we see from the recursion for ℓ_i above that if $\ell_i > k$ then $\ell_{i+1} < \ell_i$. Thus there exists some j such that $\ell_j = k$ (since $\ell_i \equiv 0 \pmod{k}$ for all i) and an easy computation shows we can assume that $j \leq \log_2(\ell_0/k) + 1$. The recursion for t_i implies that $t_i \geq t_0 2^{-i} - 1$ so that

$$t_j \geq \frac{t_0 k}{2\ell_0} - 1 \geq \frac{k(\sqrt{\ell_0 - 1} - \frac{1}{2})}{2\ell_0} - 1 \geq \frac{k}{4\sqrt{\ell_0}} \geq \frac{\sqrt{k}}{4\sqrt{D}\ln k} = D' \frac{\sqrt{k}}{\ln k}$$

(where we used that $t_0 \geq \sqrt{\ell_0 - 1} - \frac{1}{2}$, that ℓ_0 large enough since k large enough, and that $\ell_0 \leq Dk \ln^2 k$). Thus G_j has k vertices and minimum degree at least $\frac{1}{2}(k-1) + D'\sqrt{k-1}/\ln k$ and is an induced subgraph of $H[B]$ and hence an induced subgraph of G or $\overline{G}$. □

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