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Vertex-disjoint subgraphs with high degree sums

Shuya Chiba [1,2]

Department of Mathematics and Engineering, Kumamoto University, 2-39-1 Kurokami, Kumamoto 860-8555, Japan

Nicolas Lichiardopol [3]

Lycée Craponne, Salon, France

Abstract

For a graph G, we denote by $\sigma_2(G)$ the minimum degree sum of two non-adjacent vertices if G is non-complete; otherwise, $\sigma_2(G) = +\infty$. In this paper, we give the following two results; (i) If s_1 and s_2 are integers with $s_1, s_2 \geq 2$ and if G is a non-complete graph with $\sigma_2(G) \geq 2(s_1 + s_2 + 1) - 1$, then G contains two vertex-disjoint subgraphs H_1 and H_2 such that each H_i is a graph of order at least $s_i + 1$ with $\sigma_2(H_i) \geq 2s_i - 1$. (ii) If s_1 and s_2 are integers with $s_1, s_2 \geq 2$ and if G is a non-complete triangle-free graph with $\sigma_2(G) \geq 2(s_1 + s_2) - 1$, then G contains two vertex-disjoint subgraphs H_1 and H_2 such that each H_i is a graph of order at least $2s_i$ with $\sigma_2(H_i) \geq 2s_i - 1$. By using this kind of results, we also give some corollaries concerning the degree conditions for the existence of vertex-disjoint cycles.

Keywords: Vertex-disjoint, Subgraphs, Decompositions, Minimum degree sum

[1] Email: schiba@kumamoto-u.ac.jp

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[3] Email: nicolas.lichiardopol@neuf.fr

1 Introduction

In this paper, we consider finite simple graphs, which have neither loops nor multiple edges. Let G be a graph. We denote by $V(G)$, $E(G)$ and $\delta(G)$ be the vertex set, the edge set and the minimum degree of G, respectively. We denote by $d_G(v)$ the degree of a vertex v in G. The invariant $\sigma_2(G)$ is defined to be the minimum degree sum of two non-adjacent vertices of G, i.e.,

$$\sigma_2(G) = \min\Big\{d_G(u) + d_G(v) : u, v \in V(G), u \neq v, uv \notin E(G)\Big\}$$

if G is non-complete; otherwise, let $\sigma_2(G) = +\infty$. A pair (H_1, H_2) is called a *partition of* G if H_1 and H_2 are two vertex-disjoint subgraphs of G and $V(G) = V(H_1) \cup V(H_2)$.

Stiebitz [12] considered the decomposition of graphs under degree constraints and proved the following result.

Theorem 1.1 (Stiebitz [12]) *Let s_1 and s_2 be positive integers, and let G be a graph. If $\delta(G) \geq s_1 + s_2 + 1$, then there exists a partition (H_1, H_2) of G such that $\delta(H_i) \geq s_i$ for $i \in \{1, 2\}$.*

Kaneko [9] showed that this result holds for triangle-free graphs with minimum degree at least $s_1 + s_2$ as follows. (Diwan improved further Theorem 1.1 for graphs with girth at least 5, see [4].)

Theorem 1.2 (Kaneko [9]) *Let s_1 and s_2 be positive integers, and let G be a triangle-free graph. If $\delta(G) \geq s_1 + s_2$, then there exists a partition (H_1, H_2) of G such that $\delta(H_i) \geq s_i$ for $i \in \{1, 2\}$.*

The purpose of this paper is to consider σ_2-versions of Theorems 1.1 and 1.2. More precisely, we consider the following problems.

Problem 1.3 *Let s_1 and s_2 be integers with $s_1 \geq 2$ and $s_2 \geq 2$, and let G be a non-complete graph. If $\sigma_2(G) \geq 2(s_1 + s_2 + 1) - 1$, then there exists a partition (H_1, H_2) of G such that $|V(H_i)| \geq s_i + 1$ and $\sigma_2(H_i) \geq 2s_i - 1$ for $i \in \{1, 2\}$.*

Problem 1.4 *Let s_1 and s_2 be integers with $s_1 \geq 2$ and $s_2 \geq 2$, and let G be a non-complete triangle-free graph. If $\sigma_2(G) \geq 2(s_1 + s_2) - 1$, then there exists a partition (H_1, H_2) of G such that $|V(H_i)| \geq 2s_i$ and $\sigma_2(H_i) \geq 2s_i - 1$ for $i \in \{1, 2\}$.*

In Problem 1.3 (resp., Problem 1.4), if we drop the condition "$|V(H_i)| \geq s_i + 1$ (resp., $|V(H_i)| \geq 2s_i$)" in the conclusion, then it is an easy problem. Because, for each edge xy in a graph G satisfying the assumption of Problem

1.3 (resp., the assumption of Problem 1.4), $H_1 = G[\{x,y\}]$ and $H_2 = G-\{x,y\}$ satisfy $\sigma_2(H_1) = \infty > 2s_1 - 1$ and $\sigma_2(H_2) \geq \sigma_2(G) - 2|\{x,y\}| \geq 2s_2 - 1$. Here, for a vertex subset X of a graph G, $G[X]$ denotes the subgraph of G induced by X, and let $G - X = G[V(G) \setminus X]$. (Similarly, for the case where $s_i = 1$ for some i, it is easily solved.)

In addition, if G_2 is a complete bipartite graph $K_{s_1+s_2-1,s_1+s_2}$, then $\sigma_2(G_2) = 2(s_1 + s_2) - 2$ and G_2 does not contain a partition (H_1, H_2) as in Problem 1.4. Thus, G_2 shows that the condition "$\sigma_2(G) \geq 2(s_1 + s_2) - 1$" in Problem 1.4 is best possible if it's true. Moreover, if G_1 is a balanced complete multipartite graph with $l + 1$ (≥ 4) partite sets of size s (≥ 2), then $\sigma_2(G_1) = 2ls = 2((ls-l+1)+(l-1)+1)-2$, and it's easy to check that G_1 does not contain a partition (H_1, H_2) as in Problem 1.3 for $(s_1, s_2) = (ls-l+1, l-1)$. Thus the condition "$\sigma_2(G) \geq 2(s_1 + s_2 + 1) - 1$" in Problem 1.3 is also best possible in a sense if it's true.

Before giving the main result of this paper, we introduce the outline of the proof of Theorems 1.1 and 1.2. The proof of them consists of the following two steps;

Step 1: To show the existence of two vertex-disjoint subgraphs of high minimum degree, i.e., we show the existence of two vertex-disjoint subgraphs H_1 and H_2 such that $\delta(H_i) \geq s_i$ for $i \in \{1, 2\}$.
Step 2: To show the existence of two vertex-disjoint subgraphs of high minimum degree that partition $V(G)$ by using Step 1.

In particular, in the proofs of Theorems 1.1 and 1.2, Step 2 is easily solved, that is, most of the proof is Step 1. In fact, if a graph G with $\delta(G) \geq s_1+s_2-1$ contains two vertex-disjoint subgraphs H_1 and H_2 such that $\delta(H_i) \geq s_i$ for $i \in \{1, 2\}$, then we can easily extend the pair (H_1, H_2) to a partition of G keeping its minimum degree condition (see [12, Proposition 4]).

Considering the situation of the proofs of Theorems 1.1 and 1.2, one may approach to Problems 1.3 and 1.4 by the same step as above. However, for the case of σ_2-versions, Step 2 as well as Step 1 are not also an easy problem. Because we allow for graphs that there exist vertices with low degree if we consider σ_2-versions. In fact, in the proof of Step 2 for Theorem 1.1 ([12, Proposition 4]), the assumption that every vertex has high degree plays a crucial role. Although we do not know whether we can extend vertex-disjoint subgraphs of high minimum "degree sum" to a partition or not at the moment, we can solve Step 1 for Problems 1.3 and 1.4. The following are our main results.

Theorem 1.5 *Let s_1 and s_2 be integers with $s_1 \geq 2$ and $s_2 \geq 2$, and let G be a non-complete graph. If $\sigma_2(G) \geq 2(s_1 + s_2 + 1) - 1$, then there exist two vertex-disjoint induced subgraphs H_1 and H_2 of G such that $|V(H_i)| \geq s_i + 1$ and $\sigma_2(H_i) \geq 2s_i - 1$ for $i \in \{1, 2\}$.*

Theorem 1.6 *Let s_1 and s_2 be integers with $s_1 \geq 2$ and $s_2 \geq 2$, and let G be a non-complete triangle-free graph. If $\sigma_2(G) \geq 2(s_1 + s_2) - 1$, then there exist two vertex-disjoint induced subgraphs H_1 and H_2 of G such that $|V(H_i)| \geq 2s_i$ and $\sigma_2(H_i) \geq 2s_i - 1$ for $i \in \{1, 2\}$.*

Note that the above graphs G_1 and G_2 also show that σ_2 conditions in Theorems 1.5 and 1.6 are best possible, respectively.

In order to show Theorems 1.5 and 1.6, we actually consider a stronger statement as follows. Here for a graph G, we let $\varepsilon(G) = 1$ if G contains a triangle; otherwise, let $\varepsilon(G) = 0$. For a graph G and an integer s, we define $V_{\leq s}(G) = \{v \in V(G) : d_G(v) \leq s\}$.

Theorem 1.7 *Let s_1 and s_2 be integers with $s_1 \geq 2$ and $s_2 \geq 2$, and G be a non-complete graph with $\varepsilon(G) = \varepsilon$, and let $s^* = s_1 + s_2 + \varepsilon$. If $\sigma_2(G) \geq 2s^* - 1$, then there exist two vertex-disjoint induced subgraphs H_1 and H_2 of G such that for each i with $i \in \{1, 2\}$, the following hold.*

(i) *$|V(H_i)| \geq 2s_i - \varepsilon(s_i - 1)$.*

(ii) *$d_{H_i}(u) \geq s_i$ for $u \in V(H_i) \setminus V_{\leq s^*-1}(G)$.*

(iii) *$d_{H_i}(u) + d_{H_i}(v) \geq 2s_i - 1$ for $u \in V(H_i) \setminus V_{\leq s^*-1}(G)$ and $v \in V(H_i) \cap V_{\leq s^*-1}(G)$ with $uv \notin E(H_i)$.*

Note that if G is a graph with $\varepsilon(G) = \varepsilon$ and $\sigma_2(G) \geq 2(s_1+s_2+\varepsilon)-1$, then $G[V_{\leq(s_1+s_2+\varepsilon)-1}(G)]$ forms a complete graph, and hence for any two distinct non-adjacent vertices in such a graph G, at least one of the two vertices belongs to $V(G)\backslash V_{\leq(s_1+s_2+\varepsilon)-1}(G)$, i.e, (ii) and (iii) of Theorem 1.7 imply that $\sigma_2(H_i) \geq 2s_i - 1$. Thus Theorems 1.5 and 1.6 immediately follow from Theorem 1.7. Moreover, since $V_{\leq(s_1+s_2+\varepsilon)-1}(G) = \emptyset$ if and only if $\delta(G) \geq s_1 + s_2 + \varepsilon$ for a graph G with $\varepsilon(G) = \varepsilon$, Theorem 1.7 also implies Theorems 1.1 and 1.2.

In the proof of Theorem 1.7, we generalize Stiebitz's elegant argument [12] to σ_2-versions. However, the proof is rather complicated. One of the reasons is that we have to check two degree conditions ((ii) and (iii) of Theorem 1.7) to obtain the graphs H_1 and H_2 as in Theorem 1.7. One other reason is that the lower bound of the minimum degree sum of two non-adjacent vertices is not closed with respect to adding a vertex with high degree. For example, if H is a subgraph of a graph G with $\delta(H) \geq s$ and v is a vertex in $G - H$ such that v has at least s neighbors in H, then $G[V(H) \cup \{v\}]$ also has minimum degree at least s. However, it is not always true that if $\sigma_2(H) \geq 2s$ and v is a vertex in $G - H$ such that v has at least s neighbors in H, then $\sigma_2(G[V(H) \cup \{v\}])$ is also at least $2s$. Because of this, the proof of Theorem 1.7 is more difficult than Theorems 1.1 and 1.2. For more details, we refer the reader to our full version [2].

This kind of results are sometimes useful tools to get degree conditions for *packing* of graphs, i.e., the existence of k vertex-disjoint subgraphs which belong to some fixed class of graphs. In the next section, we will explain it by taking vertex-disjoint cycles for example, and give some corollaries about it.

In the rest of this section, we mention similar concepts. In 1966, Lovász [11] proved a dual type of Theorem 1.1 with respect to maximum degree; Every graph with maximum degree at most $s_1 + s_2 + 1$ has a partition (H_1, H_2) such that the maximum degree of each H_i is at most s_i. On the other hand, Thomassen [13,14] conjectured the connectivity version of Theorem 1.1; Every $(s_1 + s_2 + 1)$-connected graph has a partition (H_1, H_2) such that each H_i is s_i-connected. However, this conjecture is still wide open, and hence there is a huge gap between "minimum degree" and "connectivity". Other similar concepts can be found in [3,5,8,10,15]. Therefore, this type problem has been extensively studied in Graph Theory.

2 Applications to degree conditions for vertex-disjoint cycles

In this section, we give some corollaries for packing problems by using the results in Section 1. In particular, we will give a sharp σ_2 condition for the existence of k vertex-disjoint cycles of lengths 0-mod 3 by using Theorem 1.6 (see Problem 2.2, Proposition 2.3(ii) and Theorem 2.4).

In [1], Chen and Saito gave a minimum degree condition for the existence of a cycle of length 0-mod 3 as follows; Every graph G with $\delta(G) \geq 3$ contains a cycle of length 0-mod 3. Here, a cycle C is called a *cycle of length* 0*-mod* 3 if $|V(C)| \equiv 0 \pmod 3$. As a natural generalization of this theorem, one may consider the following problem.

Problem 2.1 *Every graph G with $\delta(G) \geq 3k$ (≥ 3) contains k vertex-disjoint cycles of lengths* 0*-mod* 3*.*

The complete bipartite graph $K_{3k-1,n-3k+1}$ shows that the minimum degree condition in Problem 2.1 is best possible if it's true, because every cycle of length 0-mod 3 in the graph has order at least 6. Considering this extremal graph, we can also consider a more general problem as follows.

Problem 2.2 *Every graph G of order at least $3k$ (≥ 3) with $\sigma_2(G) \geq 6k-1$ contains k vertex-disjoint cycles of lengths 0-mod 3.*

Since $\sigma_2(K_{3k-1,n-3k+1}) = 6k-2$, the complete bipartite graph $K_{3k-1,n-3k+1}$ also shows that "$\sigma_2(G) \geq 6k-1$" cannot be replaced by "$\sigma_2(G) \geq 6k-2$" in Problem 2.2. Moreover, since $\sigma_2(G) \geq 2\delta(G)$ for a graph G, it follows that Problem 2.2 is stronger than Problem 2.1.

In order to attack the above problems, one may use the induction on k. In particular, for Problem 2.1, we already know that Problem 2.1 is true when $k=1$ by a theorem of Chen and Saito [1], that is, Problem 2.1 can be solved by showing the inductive step. In the argument of the inductive step, Theorems 1.1, 1.2, 1.5 and 1.6 sometimes can work effectively. In fact, we can easily obtain the following by using Theorems 1.2 and 1.6, respectively.

Proposition 2.3 (i) *If Problem 2.1 is true for $k=1$, then Problem 2.1 is also true for any $k \geq 1$.*

(ii) *If Problem 2.2 is true for $k=1$, then Problem 2.2 is also true for any $k \geq 1$.*

We only show Proposition 2.3(ii) because we can obtain (i) by the same argument.

Proof. We show that Problem 2.2 is true for any $k \geq 1$ by induction on k. By the assumption of (ii), Problem 2.2 is true when $k=1$. Thus we may assume that $k \geq 2$. Let G be a graph of order at least $3k$ with $\sigma_2(G) \geq 6k-1$. We show that G contains k vertex-disjoint cycles of lengths 0-mod 3. If G is complete, then the assertion clearly holds. Thus we may assume that G is non-complete. Suppose first that G contains a triangle C. Then every vertex of G not in C has at most 3 neighbors in C, and hence $\sigma_2(G-C) \geq (6k-1)-6 = 6(k-1)-1$. Note that $|V(G-C)| \geq 3(k-1)$. Since Problem 2.2 is true for $k-1$ by the induction hypothesis, $G-C$ contains $k-1$ vertex-disjoint cycles of lengths 0-mod 3. With the cycle C, we get then k vertex-disjoint cycles of lengths 0-mod 3 in G.

Suppose next that G is triangle-free. Then, since $\sigma_2(G) \geq 6k-1 = 2(3(k-1)+3)-1$, it follows from Theorem 1.6 that there exist two vertex-disjoint subgraphs H_1 and H_2 of G such that $|V(H_1)| \geq 2\cdot 3(k-1) > 3(k-1)$, $\sigma_2(H_1) \geq 2\cdot 3(k-1)-1 = 6(k-1)-1$, $|V(H_2)| \geq 2\cdot 3 > 3$ and $\sigma_2(H_2) \geq 2\cdot 3-1 = 5$. Hence by the induction hypothesis, H_1 contains $k-1$ vertex-disjoint cycles of lengths 0-mod 3, and H_2 contains a cycle of length 0-mod 3. We get then k vertex-disjoint cycles of lengths 0-mod 3 in G. $\square$

By a theorem of Chen and Saito [1] and Proposition 2.3(i), we see that Problem 2.1 is solved in affirmative. Similarly, by Proposition 2.3(ii), it is only necessary to consider the case of $k = 1$ for Problem 2.2. In fact, we can completely solve Problem 2.2 by showing the following, which is also our main result (see also [2]).

Theorem 2.4 *Every graph G of order at least 3 with $\sigma_2(G) \geq 5$ contains a cycle of length 0-mod 3.*

Note that by the similar argument of the proof of Proposition 2.3, we can also obtain other results. For example, in [6], Gould, Horn and Magnant proposed the following conjecture, which is a generalization of Hajnal and Szemerédi's theorem [7]; Every graph G of order at least $(c+1)k$ with $\delta(G) \geq ck$ contains k vertex-disjoint cycles in which each cycle has at least $\frac{(c+1)(c-2)}{2}$ chords. They showed that the conjecture is true when c, k and the order of a graph G are sufficiently large (see [6] for more details). Moreover, they also proved that the conjecture is true when $k = 1$, and hence by combining this result with Theorem 1.1, we can easily obtain a slightly weaker version of the conjecture (see [2] for more details).

Corollary 2.5 *Let c, k be integers with $c \geq 2$ and $k \geq 1$. Every graph G with $\delta(G) \geq (c+1)k - 1$ contains k vertex-disjoint cycles in which each cycle has at least $\frac{(c+1)(c-2)}{2}$ chords.*

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