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Decomposing series-parallel graphs into paths of length 3 and triangles

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Abstract

An old conjecture by Jünger, Reinelt and Pulleyblank states that every 2-edge-connected planar graph can be decomposed into paths of length 3 and triangles, provided its size is divisible by 3. We prove the conjecture for a class of planar graphs including all 2-edge-connected series-parallel graphs. We also present a 2-edge-connected non-planar graph that can be embedded on the torus and admits no decomposition into paths of length 3 and triangles.

Keywords: 3-path decomposition, edge-decomposition, planar graph, 2-edge-connected, series-parallel

If G is a graph and $\mathcal{H} = \{H_1, \ldots, H_k\}$ is a collection of graphs, then an $\mathcal{H}$-*decomposition* of G is a collection of subgraphs of G such that each of them is isomorphic to some H_i and every edge of G is contained in exactly one of them. Let P_n and C_n denote the path and cycle on n vertices respectively. An old conjecture by Jünger, Reinelt and Pulleyblank in [2] states that every 2-edge-connected planar graph of size divisible by 3 admits a $\{P_4, C_3\}$-decomposition.

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We shall give further evidence for this conjecture by proving it for a class of graphs including all 2-edge-connected series-parallel graphs as well as for all subdivisions of prisms.

It is easy to see that all 2-edge-connected cubic graphs can be decomposed into paths of length 3, see for example [1] and [2]. By considering the dual graph it was shown by Häggkvist and Johansson in [1] that planar triangulations can be decomposed into paths of length 3. They also verified the conjecture for outerplanar graphs. Thomassen [4] showed that in general every 171-edge-connected graph can be decomposed into paths of length 3. This bound on the edge-connectivity has been pushed down to 63 in [3], and it is open whether 3-edge-connectivity is sufficient.

A graph is called *series-parallel* if it contains no K_4-minor. Every connected series-parallel graph can be constructed starting from a single edge using the following operations: subdividing an edge, replacing an edge by a pair of parallel edges, or adding a new vertex and an edge connecting it to the graph. The class of outerplanar graphs is a proper subclass of the class of series-parallel graphs. We give a short proof that 2-edge-connected series-parallel graphs have a $\{P_4, C_3\}$-decomposition, thus generalizing the main theorem in [1].

Theorem 1 *Every 2-edge-connected series-parallel graph of size divisible by 3 has a $\{P_4, C_3\}$-decomposition.*

In [1] a slightly stronger statement was proved where prepaths at two different vertices were allowed under certain constraints, which also simplified the induction. While Conjecture 2 below might seem like a strengthening of the conjecture by Jünger et al. on the first sight, it is easy to see that they are equivalent.

Conjecture 2 *If u and v are two vertices on the same face of a 2-edge-connected planar graph G, and we attach a path to u and and a path to v such that the resulting graph G' has size divisible by 3, then G' has a $\{P_4, C_3\}$-decomposition.*

Given a counterexample H to Conjecture 2, we can construct a counterexample to the original conjecture by taking three copies of H and identifying the three vertices of degree 1 on the paths attached to u, respectively v.

We prove the following strengthening of Theorem 1 and thus verify Conjecture 2 for series-parallel graphs. Notice that this also implies the result for outerplanar graphs in [1] in its full strength.

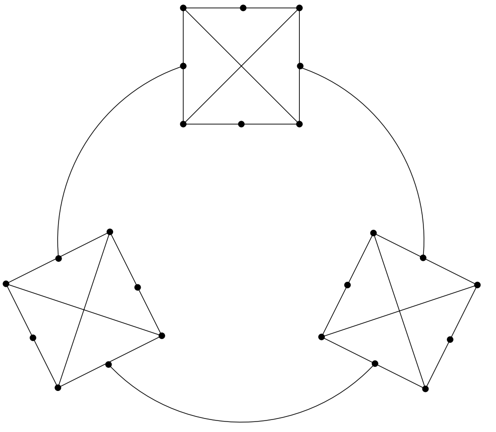

Fig. 1. A 2-edge-connected non-planar graph with no $\{P_4, C_3\}$-decomposition.

Theorem 3 *Let G be a 2-edge-connected series-parallel graph with m edges and let k and l be natural numbers such that $m+k+l$ is divisible by 3. Then for any vertices x and y in G, the graph we get by attaching a path of length k at vertex x and a path of length l at vertex y has a $\{P_4, C_3\}$-decomposition.*

Notice that it is necessary in Conjecture 2 that the paths are attached to vertices on the same face. Starting with a subdivision of K_4, we can attach two edges to two vertices not sharing a face such that the resulting graph has no $\{P_4, C_3\}$-decomposition. Hence, if the graph is not series-parallel, then it is not always possible to choose x and y arbitrarily, so Theorem 3 is in some sense best possible. It is also not possible to allow three exterior paths since a triangle with an edge attached to each vertex admits no $\{P_4, C_3\}$-decomposition. Finally, notice that it is also easy to construct a 2-edge-connected series-parallel graph of size divisible by 3 that cannot be decomposed into paths of length 3, so it is necessary to allow triangles in our decomposition.

Using the subdivision of K_4 with two additional edges as a building stone, we construct a 2-edge-connected graph that admits no $\{P_4, C_3\}$-decomposition, see Figure 1. This example is smaller than the one given in [2] and can also be embedded on the torus, unlike the previous example.

We also use Theorem 3 to construct a larger class of graphs with a $\{P_4, C_3\}$-decomposition. Let e be an edge of a graph G and let H be a 2-edge-connected series-parallel graph with two specified vertices x and y. We say that G' is constructed from G by *replacing e by H* if G' is formed from the disjoint union of $G-e$ and H by identifying x and y with distinct endpoints of e. We call such an operation an *edge-replacement*. Let $\mathcal{G}_0$ denote the class of all 2-edge-connected series-parallel graphs with up to two paths attached to it. Note that a graph in $\mathcal{G}_0$ need not be series-parallel.

Corollary 4 *If G can be constructed from a graph in $\mathcal{G}_0$ by finitely many edge-replacements, and G has size divisible by 3, then G has a $\{P_4, C_3\}$-decomposition.*

The conjecture by Jünger et al. can be reduced to planar subcubic graphs by splitting vertices of degree larger than 3 such that the resulting graph is still planar and 2-edge-connected. Thus it is of interest to check whether all subdivisions of a given cubic graph admit a $\{P_4, C_3\}$-decomposition. The k-*prism* is defined as the cartesian product of C_k and P_2. As an application of Theorem 3 we get the following corollary.

Corollary 5 *Every subdivision of a k-prism ($k \geq 3$) can be decomposed into paths of length 3, provided the size is divisible by 3.*

References

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