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The Multicolour Ramsey Number of a Long Odd Cycle

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Abstract

For a graph G, the k-colour Ramsey number $R_k(G)$ is the least integer N such that every k-colouring of the edges of the complete graph K_N contains a monochromatic copy of G. Bondy and Erdős conjectured that for an odd cycle C_n on $n > 3$ vertices,

$$R_k(C_n) = 2^{k-1}(n-1) + 1.$$

This is known to hold when $k = 2$ and $n > 3$, and when $k = 3$ and n is large. We show that this conjecture holds asymptotically for $k \geq 4$, proving that for n odd,

$$R_k(C_n) = 2^{k-1}n + o(n) \text{ as } n \to \infty.$$

The proof uses the regularity lemma to relate this problem in Ramsey theory to one in convex optimisation, allowing analytic methods to be exploited. Our analysis leads us to a new class of lower bound constructions for this problem, which naturally arise from perfect matchings in the k-dimensional hypercube. Progress towards a resolution of the conjecture for large n is also discussed.

Keywords: Ramsey number, regularity lemma, convex optimisation, hypercube.

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1 Introduction and Results

For a graph G, the k-colour Ramsey number $R_k(G)$ is the least integer N such that every k-colouring of the edges of the complete graph K_N contains a monochromatic copy of G. We let C_n denote the cycle of length n. The Ramsey numbers of cycles has been the subject of much study, in particular the value of $R_2(C_n)$ has been determined for all n through the work of Bondy and Erdős [1], Faudree and Schelp [4], and Rosta [8]. For more than two colours the problem is far less understood. The following conjecture is attributed to Bondy and Erdős [1].

Conjecture 1 *If $k \geq 3$ and $n > 3$ is odd, then*

$$R_k(C_n) = 2^{k-1}(n-1) + 1.$$

We note that the value of $R_k(C_n)$ when n is even exhibits different behaviour where the conjectured value is $(k-1)n + O(1)$ as $n \to \infty$.

For $k \geq 3$ and odd $n > 3$, Erdős and Graham [3] proved the bounds $2^{k-1}(n-1)+1 \leq R_k(C_n) \leq (k+2)!n$. The lower bound motivates Conjecture 1, which the authors establish with a simple inductive construction: If there exists a k-colouring of the edges of the complete graph K_m with no monochromatic C_n, then by joining two such copies of K_m by edges of colour $k+1$, one obtains a $(k+1)$-colouring of K_{2m} with no monochromatic C_n. The base construction, for $k = 1$, is simply a monochromatic clique of size $n-1$.

The first breakthrough towards Conjecture 1 was made by Łuczak [6] who used the regularity method to show that the $k = 3$ case holds asymptotically, i.e. that for n odd,

$$R_3(C_n) = 4n + o(n) \text{ as } n \to \infty.$$

More recently, Kohayakawa, Simonovits and Skokan [5] paired Łuczak's approach with stability arguments to resolve the $k = 3$ case of Conjecture 1 for large n. The case where $k \geq 4$ remains open. Progress was made by Łuczak, Simonovits and Skokan [7] who showed that for $k \geq 4$ and odd n, $R_k(C_n) \leq k2^k n + o(n)$ as $n \to \infty$. In this paper we show that Conjecture 1 holds asymptotically for all k.

Theorem 1 *For $k \geq 4$ and odd n,*

$$R_k(C_n) = 2^{k-1}n + o(n) \text{ as } n \to \infty.$$

The proof of Theorem 1, which we sketch in the following section, leads us to a new class of extremal colourings for Conjecture 1 which arise naturally from perfect matchings of the k-dimensional hypercube Q_k. We will formulate a conjecture, Conjecture 2, which asserts that these colourings are essentially the only extremal colourings for Conjecture 1. We discuss progress towards this conjecture and towards a resolution of Conjecture 1 for large n.

2 Proof Methods and Hypercube Colourings

For the proof of Theorem 1, let n be odd, $\epsilon > 0$ and set $N = 2^{k-1}n + \epsilon n$. Suppose that there exists a k-colouring of the edges of $G = K_N$, avoiding a monochromatic copy of C_n. Let $G_1, \ldots, G_k$ be its colour classes. We apply the k-colour version of the regularity lemma [9], with a suitable choice of parameters, to obtain a regular partition of the vertex set $V(G)$ into $t+1$ classes $V(G) = V_0 \cup \ldots \cup V_t$. We construct a reduced graph R with vertex set $1, ..., t$ and the edge set formed by pairs $\{i, j\}$ for which (V_i, V_j) is regular with respect to $G_1, \ldots, G_k$. We k-colour R by the majority colour in the pair (V_i, V_j). The crucial point is that the graph R cannot contain a monochromatic, non-bipartite, connected subgraph with a matching of size greater than $t/2^k$ since that would imply the existence of a monochromatic copy of C_n in the original graph G (see [6], [7]). The following theorem of Erdős and Gallai [2] shows that forbidding a large matching in each connected component of a graph is very restrictive, in particular one forbids long cycles.

Theorem 2 *Let $m \geq 2$. If G is a graph such that G contains no cycle of length greater than m, then $e(G) \leq m(v(G)-1)/2$.*

We begin with a decomposition of R similar to the one exploited in [7]. Let $R_1, \ldots, R_k$ be the colour classes of R. We may write $R_i = R_i' \cup R_i''$, where R_i' is the union of the bipartite components of R_i and R_i'' is the union of the non-bipartite components of R_i. We now classify the vertices of R according to their position in this partition for each colour. For $i \in [k]$, write $V(R_i') = V_0^i \cup V_1^i$ where V_0^i and V_1^i are the vertex classes of a bipartition of R_i' and set $V_*^i = V(R_i'')$. For $\tau = (\tau_1, \ldots, \tau_k) \in \{0, 1, *\}^k$, let $V_\tau = \bigcap_{j=1}^k V_{\tau_j}^j$ and note that $V(R) = \bigcup_{\tau \in \{0,1,*\}^k} V_\tau$, a disjoint union.

The main idea of the proof is now readily explained. We may think of an element $\tau \in \{0, 1, *\}^k$ as a subcube of the k-dimesional hypercube Q_k via the correspondence $\tau \to Q(\tau)$ where $Q(\tau) = \{c \in \{0, 1\}^k : c_j = \tau_j \text{ whenever } \tau_j \in \{0, 1\}\}$. In other words we think of a coordinate whose entry is $*$, as a 'missing bit' and we consider the set of all possible ways of filling in these bits. In particular if τ

has only one bit missing then we think of $Q(\tau)$ as an *edge* of Q_k in the natural way. Let $m = 3^k$, let $\{\tau_1, \dots, \tau_m\}$ be a fixed enumeration of the elements of $\{0, 1, *\}^k$ and set $v = (v_1, \dots, v_m) := (|V_{\tau_1}|, \dots, |V_{\tau_m}|)$. Using Theorem 2 to control the density of edges in each colour between the parts $V_{\tau_1}, \dots, V_{\tau_m}$ of the graph R, one obtains an inequality of the form

$$F(v) \leq 0, \tag{1}$$

where F is a quadratic form that we do not specify here. We then view (1) as a constraint in a non-linear programme where we wish to maximise $v(R) = \sum_{i=1}^{m} v_i$, viewed as the objective function. This analytic viewpoint allows us to import tools from the theory of convex optimisation where first we use the combinatorial technique of 'compression' or 'shifting' to reduce (1) to a spherical constraint. What is remarkable is that (1) is strong enough to imply that $v(R) = \sum_{i=1}^{m} v_i < t$ contradicting the assumption that $v(R) = t$ and thus proving Theorem 1.

Looking more closely at the optimisation problem discussed above, one finds that $\sum_{i=1}^{m} v_i$ is maximised precisely when v is supported on a perfect matching of the hypercube Q_k (recall that the coordinates of v are indexed by the subcubes of Q_k). These optimal points naturally correspond to the following colourings of K_M, where $M = 2^{k-1}(n-1)$.

Let $\mathcal{M}$ be a perfect matching of Q_k. We express each edge of $\mathcal{M}$ as an element of $\{0, 1, *\}^k$. For each edge $\tau \in \mathcal{M}$ form a monochromatic clique $K(\tau)$ of size $n - 1$ and colour i, where i is the coordinate for which $\tau_i = *$. For $\tau, \sigma \in \mathcal{M}$, the edges between $K(\sigma)$ and $K(\tau)$ can be arbitrarily coloured with any colour j for which τ, σ lie in opposite subcubes of Q_k of codimension 1 separated by the jth coordinate (i.e. either $\sigma_j = 0, \tau_j = 1$ or $\sigma_j = 1, \tau_j = 0$). It is not hard to show that such a colouring avoids monochromatic copies of C_n. Let us call such a colouring a *hypercube colouring*. If we inductively construct a perfect matching on Q_k by taking two perfect matchings on a disjoint pair of subcubes of codimension 1 and consider the associated hypercube colouring, we recover the colourings considered by Erdős and Graham [3] that we described in the Introduction. However for $k \geq 4$, not all perfect matchings of Q_k decompose as the union of two matchings on a pair of codimension 1 subcubes, and so we obtain some genuinely new colourings. We conjecture that the following stability result holds. Recall that for a k-coloured graph G, we let G_i denote its ith colour class.

Conjecture 2 *Let $k \geq 4$. Then for any $\epsilon > 0$, there exists an $\eta > 0$ and $n_0 \in \mathbb{N}$, such that for all odd $n > n_0$ and $N > (2^{k-1} - \eta)n$ the following holds. If*

$G = K_N$ *is* k*-coloured with no monochromatic copy of* C_n*, then* $N \leq 2^{k-1}(n-1)$ *and there exists a hypercube colouring of the complete graph on* $2^{k-1}(n-1)$ *vertices* H *such that* $V(G) \subseteq V(H)$ *and* $|G_i \triangle H_i| \leq \epsilon N^2$ *for all* $i \in [k]$*.*

The $k = 3$ case of Conjecture 2 was proved in [5] where the two classes of colourings the authors consider can be viewed as the colourings that arise from the two isomorphism classes of perfect matchings in Q_3.

A proof of Conjecture 2 is a work in progress. Our starting point is an 'analytic stability' statement asserting that an almost optimal point of the aforementioned non-linear programme must be very close to an optimal point in Euclidean distance.

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