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Triangles in random cubic planar graphs

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Abstract

In this extended abstract we determine a normal limiting distribution for the number of triangles in a uniformly at random 3-connected cubic planar graph, as well as the precise expectation and variance values. Further comments towards the more complicated problem of studying both the limiting distribution of triangles in random cubic planar graphs, and the (asymptotic) number of triangle-free cubic planar graphs are discussed as well.

Keywords: random planar graph, analytic combinatorics, limiting distributions.

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1 Introduction

The theory of random graphs, initiated by Erdős and Rényi in their seminal paper *On the evolution of Random Graphs* in the early 1960s [9], has become one of the main areas of research in combinatorics. Nowadays, there is a rich theory which explains this model, and a wide range of parameters are well understood (see for instance [5,17]). Unfortunately, the analysis becomes more involve if one wants to analyse random graphs subject to a global condition, such as avoiding a certain pattern. In this extended abstract we deal with graphs subject to planar obstructions. The first systematic study of these type of objects (in an embedded setting) was addressed by Tutte in the sixties [20,21,22] with the objective of getting an enumerative solution of the four colours problem.

Later, models of random planar graphs were introduced and studied as well (see, for instance [7,19]). Lot of activity has been devoted to the comprehension of this family of random graphs. In this context, analytic methods based on generating functions and complex analytic tools over them have been developed dramatically in the last years to handle more and more complicated classes of graphs. The asymptotic enumeration of the number of labeled planar graphs on n vertices by Giménez and Noy [14] (building on a previous work of Bender, Gao and Wormald on 3-connected (and 2-connected) planar graphs [2]) was a breakthrough on the use of these techniques. Since then, some subfamilies of planar graphs have been studied as well, and even graphs on surfaces (see [6]).

In particular, nowadays many results are known when dealing with pending copies of a given subgraph, including planar graphs [14], series-parallel graphs [3] and more generally for graphs defined by 3-connected components [15] (see also [18] for minor-free closed families of graphs). In most of the cases, the corresponding limiting distribution satisfies a central limit law. Unfortunately, the more complicated problem of studying copies of a given subgraph is still widely open in the general setting (see [8] for results on trees).

In this work, we study families of planar graphs with *subgraph obstructions* by using analytic techniques (see the reference books [10,8]). In particular, we are interested in the following question: given a subgraph H, how many subgraphs isomorphic to it are there in a uniformly at random graph in the class under consideration? And how many graphs are there in the family which are H-free? See for instance [12,13] for works on this direction. These questions are made explicit by studying the subgraph $H = K_3$ on a uniformly at random 3-regular planar graph. Our project extends previous results of Gao and Wormald who obtained counting formulas for 3-connected triangle-free cubic planar maps. Let us mention that the enumerative problem (namely, the asymptotic enumeration of labeled cubic planar graphs with a fixed number of vertices) was solved by Bodirsky, Kang, Löffler and McDiarmid in [4]. This result (as explained in Section 5) will be crucial to complete the analysis of the full family in the full version of the paper.

This extended abstract is divided as follows: in Section 2 we introduce the basic definitions needed on graphs and maps enumeration, as well as for generating functions. Later, in Section 3 we get the desired generating functions, and in Section 4 we analyse them by means of analytic tools. Finally, in Section 5 we discuss the next steps we expect to develop in the full version of the work.

2 Preliminaries: Rooted maps and generating functions

In this abstract, all graphs are undirected, simple and loopless. A *labeling* of the n vertices of a graph G is a bijection $\ell : V(G) \to \{1, \ldots, n\}$. A graph is said to be *labeled* when it is given together with a labeling of its vertices. A *planar map*, or *map* for short, is a connected graph embedded on the *sphere* $\mathcal{S}^2$ without edge crossing. It is said to be *rooted* when we choose an edge, the *root edge*, a vertex from this edge, the *root vertex*, and one of the two faces adjacent to the root edge, the *root face*. Observe that there are $4e$ different ways to root a map with e edges and with trivial automorphism group. An edge or a vertex of the map is said to be *external*, when it is adjacent to the root face, and *inner* otherwise. In the following, rooted maps will be considered up to the equivalence class given by homeomorphisms preserving the sphere and its orientation.

A map is said to be k-connected, for a positive integer k, when the removal of any subset of at most $k-1$ vertices does not disconnect the map. When dealing with 3-connected graphs, Whitney's celebrated theorem guarantees that any two planar embeddings of a 3-connected graph are equivalent up to homeomorphism of the sphere (see [23]). Hence, instead of studying 3-connected labeled planar graphs, we can restrict ourselves to the study of rooted 3-connected planar maps.

In a map, a *triangle* is a face of degree three together with its boundary. A map is said to be *cubic* when all its vertices have degree three, and *triangular* when every face, with its boundary, is a triangle. Observe that the dual of a cubic map is a triangular map, because cubic vertices become triangles. Consequently, there are as many 3-connected cubic maps as there are 3-connected triangular maps. So, in order to count triangles in 3-connected cubic maps, it is sufficient to count cubic vertices in 3-connected triangular maps.

We study enumerative problems on these type of objects. We use ordinary generating functions to encode enumerative counting formulas for maps, while we use exponential generating functions when dealing with labeled graphs.

3 Counting formulas for maps and 3-connected graphs

Given an integer k, we define a t_k-type map as a k-connected triangular rooted map with at least $k+1$ vertices. Let then $t_k(x,u)$ be the generating function counting t_k-type maps where x denotes the total number of vertices minus two and u the number of inner cubic vertices. For example, the t_3-type map K_4 is denoted by x^2u. With this notation, the enumeration of cubic vertices in 3-connected rooted maps is given by the coefficients of the variable u in $t_3(x,u)$. When $u=1$, we will use the short notation $t_k(x)$ instead of $t_k(x,1)$. Tutte in its seminal paper on the enumeration of planar triangulations [20] already studied t_4-type maps and gave the following parametrization for it, that will be our starting point:

$$t_4(z) = z + \frac{t(t+1)}{(1+t)^2} - z^2, \tag{1}$$

$$z = t(1-t)^2. \tag{2}$$

Indeed, computing the coefficients for the cubic vertices will be done following a decomposition used in [11], which is inspired by the work of Tutte [20]. This decomposition works by pasting t_3-type maps inside the inner faces of a *core*, a t_4-type map or K_4. This substitution is formalized via the composition of generating functions. Observe that, when doing such substitutions, the only problematic situation, concerning the coefficients of the variable u, arises when the core is K_4. Hence, we distinguish two cases: one for when the core is K_4, and one for when it is a t_4-type map. Finally, by adapting the variable x to count inner faces instead of vertices minus two, we obtain the following functional equation for t_3-type maps:

Lemma 3.1

$$t_3(x,u) = \frac{t_4\Big(x(1+x^{-1}t_3(x,u))^2\Big)}{1+x^{-1}t_3(x,u)} + x^2\Big(1+x^{-1}t_3(x,u)\Big)^3 + x^2(u-1).$$

We now define $T_3(x,u)$, the generating function whose coefficients count the number of 3-connected rooted triangular maps, where x still counts the number of vertices minus two, but u counts now the total number of cubic vertices. Observe that in this case, K_4 becomes x^2u^4. We have the following lemma.

Lemma 3.2 *The following equality holds*

$$T_3(x,u) = t_3(x,u)\Big(1 + 3x(u-1)\Big) + x^2(u^4 - u). \tag{3}$$

We sketch a proof of this result. To enumerate T_3-type maps, we proceed the same way as in the previous lemma, by substituting t_3-type maps in inner faces of a t_4-type map or a K_4. Hence,

$$\begin{aligned} T_3(x,u) = & \frac{t_4\Big(x(1 + x^{-1}t_3(x,u))^2\Big)}{1 + x^{-1}t_3(x,u)} + x^2u^4 \\ & + x^2\Big(1 + x^{-1}t_3(x,u)\Big)^3 - x^2 + 3x^2(u-1)x^{-1}t_3(x,u), \end{aligned}$$

where substitutions on t_4-type maps work exactly the same as before and substitutions on K_4 are divided now in three different cases, given the number of inner faces actually substituted:

(i) no inner face gives the term x^2u^4,

(ii) exactly one inner face gives the term $3x^2(u-1)x^{-1}t_3(x,u)$,

(iii) and two or three inner faces give the term $x^2(1 + x^{-1}t_3(x,u))^3 - x^2$.

Then, using Lemma 3.1, we can eliminate t_4 from the previous equation. Which gives us, after simplification, the functional equation of the theorem. Let us mention that the case $u = 0$ (triangle-free 3-connected rooted cubic planar maps) was studied already by Gao and Wormald in [11].

Finally, we proceed with the two final stages, namely building by duality the generating function $M_3(x,u)$, where u counts triangles into 3-connected rooted cubic maps and x marks vertices, and then finally translating to the labeled planar graph setting, into the generating function $G_3(x,u)$ counting triangles in 3-connected cubic labeled planar graphs.

The first step is done by just applying Euler's relation, which gives us that $M_3(x,u) = T_3(x^2,u)$. For the second step, we need to transform an unlabeled rooted planar map into an unrooted labeled planar graph. Suppose that $m_{n,t}$ is the number of 3-connected rooted cubic maps with n vertices and t triangles, and $g_{n,t}$ is the number of 3-connected cubic n-vertex labeled planar graphs, also with t triangles. We then have $n!$ different ways to label the vertices of a n-vertex map and $4e$ ways to root a labeled graph with e edges (because of their 3-connectivity, their automorphism group is trivial). And because we consider cubic graphs, by a double-counting argument we have that $e = \frac{3}{2}n$. Hence, $6n \cdot g_{n,t} = n! \cdot m_{n,t}$, and

$$G_3(x,u) = \sum_{n,t} \frac{g_{n,t}}{n!} x^n u^t = \sum_{n,t} \frac{1}{6n} m_{n,t} x^n u^t = \frac{1}{6} \int_0^x \frac{T_3(s,u)}{s} ds.$$

4 Asymptotic enumeration and limit laws

The first consequence of the previous counting formulas is the following limiting estimate for the number of triangles in a uniformly at random 3-connected cubic planar graph:

Theorem 4.1 *Let X_n be the number of triangles in a uniformly at random 3-connected cubic planar graph on $2n$ vertices. Denoting by μ_n and σ_n^2 the expectation and variance of X_n, we have that*

$$\frac{X_n - \mu_n}{\sigma_n} \to N(0,1),$$

where $N(0,1)$ is the standard normal distribution. Additionally, the mean μ_n and the variance σ_n^2 satisfy

$$\mu_n = \frac{27}{128} n(1+o(1)),\ \sigma_n^2 = \frac{3267}{32768} n(1+o(1)).$$

We sketch the proof of this theorem. For each choice of u (and v), observe that $T_3(x,u)$ is an analytic transform of $t_3(x,v)$ (see Equation (3)), hence the position (and nature) of the singularities of both generating functions are the same. Similarly, $G_3(x,u)$ is obtained by integration of $T_3(x,u)$. Hence, the singularities of $G_3(x,u)$ are the same as $t_3(x,u)$ (with the difference that now the singularity type is different). So, all the analysis could be done over $t_3(x,u)$.

Using Equations (1),(2), by setting $x(1+x^{-1}t_3(x,u))^2 = t(1-t)^2$, and elimination theory packages from `Maple`, we get that $t_3 := t_3(x,u)$ satisfies the following implicit equation of degree 4 in t_3:

$$0 = 8x^5 - 21x^6u^2 - 3x^5u - x^2u + x^7u^3 + 18x^6u + 3x^7u - 26x^4u - 5x^6 - 3x^7u^2 + 8x^6u^3 + 16x^5u^3 - x^7 - x^4 + 28x^4u^2 + 11x^3u - 21x^5u^2 + (24x^5u + 57x^4u^2 - 102x^4u + 16x^5u^3 - 20x^3u^2 + 59x^2 + 12x^6u + 106x^3u - 12x^6u^2 + 45x^4 - 14x - 4x^5 + 1 - 36x^5u^2 - 4x^6 - 34x^2u - 82x^3 + 4x^6u^3)t_3 + (8x^3u^2 + 6x^4u^2 - 12x^4u - 4x^3u + 25x^2u + 6x^4 - 4x^3 - 19x^2 + 17x + 3)(xu - x + 1)t_3^2 + (4x^2u + 3 + 4x - 4x^2)(xu - x + 1)^2 t_3^3 + (xu - x + 1)^3 t_3^4.$$

Hence, $t_3(x,u)$ satisfies an implicit (algebraic) equation of the form $0 = P(t_3,x,u)$, where $P(T,X,U)$ is a polynomial with integer coefficients. For each fixed u, the points where $t_3(x,u)$ ceases to be analytic are the ones satisfying the system of equations

$$0 = P(T,X,U),\ 0 = P_T(T,X,U).$$

Computation again with `Maple` shows that the system of three equations $0 = P(T,X,U)$, $0 = P_T(T,X,U)$, $0 = P_{TT}(T,X,U)$ does not have a solution when u varies in a neighborhood of $u = 1$. This fact shows that the singularity type of $t_3(x,u)$ around $x(u)$, solution of the system $0 = P(T,X,U)$, $0 = P_T(T,X,U)$, is of square-root type independent of the choice of u. In this context, one can apply the celebrated Quasi-Powers Theorem by Hwang (see [16]), which assures normal limiting distributions when the singularity type of the generating function under study does not change when u varies around 1. In particular, the expectation and variance are both linear, with constants

$$\mu = -\frac{x'(1)}{x(1)},\ \sigma = -\frac{x''(1)}{x(1)} - \frac{x'(1)}{x(1)} + \left(\frac{x'(1)}{x(1)}\right)^2.$$

This computation can be done by exploiting the previous implicit equation: eliminating from equations $0 = P(T,X,U)$, $0 = P_T(T,X,U)$ the variable T, we get the curve of singularities of x in terms of u:

$$256x(xu - x + 1)^2 - 27 = 0. \tag{4}$$

In particular, for $v = 1$, the singular point arises at $x(1) = \frac{27}{256}$. This equation defines implicitly x in terms of u. Hence, we can compute, both $x'(u)$ and $x''(u)$, in terms of $x(u)$ only. The estimates for μ and σ are obtained by computing (by means of Equation (4)) $x'(1)$ and $x''(1)$. In this case, computations give $x'(1) = -\frac{729}{32768}$ and $x''(1) = \frac{137781}{8388608}$.

5 Further research

In this extended abstract we obtain a first result on limiting distributions for the number of triangles in a planar-like family of graphs (namely, graphs arising from 3-connected maps in the sense of [15]). We expect to extend this study to general random cubic planar graphs by refining the equations developed by Bodirsky, Kang, Löffler and McDiarmid in [4]. Indeed, the methods developed so far, joint with this extension of the results in [4] and the particularities of dealing with cubic graphs would provide also limiting distributions for the number of subgraphs isomorphic to K_4^- (namely, K_4 minus an edge). In both cases (number of triangles, and number of K_4^-) we expect that the limiting distribution will be of normal type with linear expectation and variance. Asymptotic formulas for triangle-free cubic planar graphs on vertices would be explored as well.

Complementarily to these statistics and these counting formulas, we expect to extend the investigations in [4] with the purpose of studying the size of the largest 3-connected component in a random cubic planar graph. The main tools that will be used at this part of the investigation will be the Airy distributions arising from critical schemes in map families [1], joint with the analogue results in planar-like families obtained in [15].

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