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Triangle-Free Subgraphs of Random Graphs

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Abstract

The Andrásfai–Erdős–Sós Theorem [2] states that all triangle-free graphs on n vertices with minimum degree strictly greater than $2n/5$ are bipartite. Thomassen [11] proved that when the minimum degree condition is relaxed to $(\frac{1}{3}+\varepsilon)n$, the result is still guaranteed to be r_ε-partite, where r_ε does not depend on n. We prove best possible random graph analogues of these theorems.

Keywords: Random Graphs, Sparse Regularity, Andrásfai–Erdős–Sós Theorem, Chromatic Threshold

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1 Introduction

In a 1948 edition of the recreational math journal Eureka, Blanche Descartes proved that triangle-free graphs can have arbitrarily large chromatic number, and thus be complex in structure. This motivates the question of which additional restrictions on the class of triangle-free graphs allow for a bound on the chromatic number. By Mantel's theorem [10], the densest triangle-free graphs are balanced complete bipartite graphs. So we may first ask whether triangle-free graphs H with minimum degree somewhat below $\frac{1}{2}v(H)$ are still necessarily bipartite. This is true, as Andrásfai, Erdős and Sós showed in 1974.

Theorem 1.1 (Andrásfai, Erdős, Sós [2]) *All triangle-free graphs H with $\delta(H) > \frac{2}{5}v(H)$ are bipartite.*

Triangle-free graphs of smaller minimum degree do not need to be bipartite, as blow-ups of a 5-cycle illustrate. But one may still ask whether their chromatic number is bounded (questions of this type were first addressed by Erdős and Simonovits in [7]). In 2002 Thomassen [11] proved that this is the case for triangle-free graphs of minimum degree at least $(\frac{1}{3}+\varepsilon)n$.

Theorem 1.2 (Thomassen [11]) *For any $\varepsilon > 0$ there exists r_ε such that if H is triangle-free and $\delta(H) > (\frac{1}{3}+\varepsilon)v(H)$ then H is r_ε-colourable.*

A construction of Hajnal (see [7]) shows that the minimum degree bound in this theorem cannot be replaced by $(\frac{1}{3}-\varepsilon)n$. A much stronger result was established by Brandt and Thomassé [3], who showed that triangle-free graphs H with $\delta(H) > \frac{1}{3}n$ are 4-colourable.

In this paper we are interested in random graph analogues of Theorem 1.1 and Theorem 1.2. Establishing such analogues for prominent results in extremal graph theory has been a particularly fruitful area of study in the last few years. A good overview can be found in Conlon's survey paper [4].

In order to study these kinds of questions systematically, Kohayakawa [8] and Rödl (unpublished) developed a sparse analogue of Szemerédi's Regularity Lemma, and, together with Łuczak [9] formulated the KŁR conjecture which asserts the existence of a corresponding 'counting lemma'. Recently Conlon, Samotij, Schacht and Gowers [5] proved this conjecture. It is easy (as observed in [5]) to use these results to prove 'approximate' random versions of Theorems 1.1 and 1.2, as well as re-prove Mantel's theorem for random graphs. Thus if $p \gg n^{-1/2}$ then a.a.s. $G(n,p)$ has the property that all subgraphs with minimum degree a little larger than $\frac{2}{5}pn$ can be made bipartite by deleting $o(pn^2)$ edges. Similarly, the sparse random version of Mantel's theorem obtained states that any subgraph with a little more than half the edges of $G(n,p)$ contains a triangle.

At first it might seem surprising that there are subgraphs of $G_{n,p}$ with minimum degree a little larger than $\frac{2}{5}pn$ which are not actually bipartite. Indeed, an alternative sparse random version of Mantel's theorem, proved by DeMarco and Kahn [6], states that a largest triangle-free subgraph of $G(n,p)$ coincides exactly with a largest bipartite subgraph for $p \gg (\frac{\log n}{n})^{1/2}$. Nevertheless, subgraphs of $G(n,p)$ with minimum degree larger than $\frac{2}{5}pn$ which are not bipartite do exist (see Theorem 1.5 below). In this paper, we determine, for all p, how far from bipartite such graphs can be.

Theorem 1.3 *For any $\gamma > 0$, there exists C such that for any $p(n)$ the random graph $\Gamma = G(n,p)$ a.a.s. has the property that all triangle-free spanning subgraphs $H \subseteq \Gamma$ with $\delta(H) \geqslant (\frac{2}{5} + \gamma)pn$ can be made bipartite by removing at most* $\min\left(Cp^{-1}n, (\frac{1}{4} + \gamma)pn^2\right)$ *edges.*

In addition we derive an analogous random graph version of Theorem 1.2.

Theorem 1.4 *For any $\gamma > 0$, there exist C and r such that for any $p(n)$ the random graph $\Gamma = G(n,p)$ a.a.s. has the property that all triangle-free spanning subgraphs $H \subseteq \Gamma$ with $\delta(H) \geqslant (\frac{1}{3} + \gamma)pn$ can be made r-partite by removing at most* $\min\left(Cp^{-1}n, (\frac{1}{2r} + \gamma)pn^2\right)$ *edges.*

Up to the values of C, these theorems are best possible.

Theorem 1.5 *For any $\gamma > 0$ and $r \in \mathbb{N}$, there exist constants $\delta, c > 0$ such that if $\delta^{-1}n^{-1/2} \leqslant p(n) \leqslant \delta$ then $\Gamma = G(n,p)$ a.a.s has a triangle-free spanning subgraph H with $\delta(H) \geqslant (\frac{1}{2} - \gamma)pn$ which cannot be made r-partite by removing fewer than $cp^{-1}n$ edges.*

Note that for $p \ll n^{-1/2}$ the maximum in each of Theorems 1.3 and 1.4 is achieved by the second term and that these statements are easy: For such values of p only a tiny fraction of the edges of $G(n,p)$ are in triangles and the question reduces to asking for the largest bipartite (respectively, r-partite) subgraph of $G(n,p)$. For p close to 1, by the original Theorems 1.1 and 1.2, the conclusion of Theorem 1.5 becomes false, so that we need the condition $p \leqslant \delta$.

It would be interesting to know whether Theorem 1.4 could be improved to generalise the result of Brandt and Thomassé. We conjecture that this is the case.

2 Proof outline

We first sketch the construction proving Theorem 1.5, and then outline the proof of Theorem 1.3. The proof of Theorem 1.4 uses similar ideas.

For Theorem 1.5, let $\Gamma = G(n,p)$, and let $X \cup Y$ be a balanced bipartition of $V(\Gamma)$. We construct $G \subseteq \Gamma$ as follows. We first delete all edges in Y. We next randomly delete edges in X until $cp^{-1}n$ edges remain in X. Finally, we delete all edges within X which are in triangles contained in X, and all edges between X and Y which are in triangles. Then G is clearly triangle-free. Moreover, $G[X]$ has large chromatic number because most edge deletions in X were done randomly. With some care, it is also possible to show that the maximum cut in G is $X \cup Y$.

To prove Theorem 1.3, we work in two stages. In the first step, we show that given a triangle-free $G \subseteq \Gamma = G(n,p)$ with $\delta(G) \geqslant (\frac{2}{5}+\gamma)pn$, a maximum cut of $V(G)$ has at most $o(pn^2)$ edges within its parts. This we can do using the sparse Regularity Lemma and the solution to the KŁR conjecture. In the second step, using this rough structure, we argue that in fact there can only be $Cp^{-1}n$ edges within the parts. This is the novel part of the argument. Suppose that $G[X]$ contains more than $\frac{C}{2}p^{-1}n$ edges. (Almost) all of these edges extend to about $p^2|Y|$ triangles in Γ with vertices of Y (these triangles are of course not present in G). It follows that edges between X and Y in Γ must have been deleted to obtain G. We give an orientation to each edge of $G[X]$, towards the endpoint at which more X–Y edges were deleted. Now in this oriented graph, an in-star (a star with all edges oriented to the centre) with many leaves corresponds to many deletions of X–Y edges made at the centre vertex v. If all these deleted edges were distinct, then we would conclude that v has less than $\frac{1}{5}pn$ neighbours in Y, and (by maximality of the cut) less than $\frac{2}{5}pn$ neighbours in total, a contradiction which would complete the proof. Unfortunately, the triangles of G at v may overlap and hence the deletions corresponding to the edges of the in-star may not be distinct. Nevertheless, we can show that given many disjoint such in-stars there must be one centre where the overlaps are small, and that given many edges in $G[X]$ there must be many disjoint such in-stars. This allows us to complete our proof. A technical difficulty here is that we must treat vertices of X with many neighbours in X separately.

The proof of Theorem 1.4 is similar. The main conceptual difference is that we use a 'regularity inheritance' lemma from [1] to help obtain the initial rough structure, before using an 'in-stars' argument to complete the proof.

References

[1] Allen, P., J. Böttcher, H. Hàn, Y. Kohayakawa and Y. Person, *Blow-up lemmas for sparse graphs* (2015), in preparation.

[2] Andrásfai, B., P. Erdős and V. T. Sós, *On the connection between chromatic number, maximal clique and minimal degree of a graph*, Discrete Math. **8** (1974), pp. 205–218.

[3] Brandt, S. and S. Thomassé, *Dense triangle-free graphs are four colorable: A solution to the Erdős-Simonovits problem*, to appear in J. Combin. Theory Ser. B.

[4] Conlon, D., *Combinatorial theorems relative to a random set*, proceedings of the 2014 ICM, to appear.

[5] Conlon, D., T. Gowers, W. Samotij and M. Schacht, *On the KŁR conjecture in random graphs* (2013), israel J. Math, to appear.

[6] DeMarco, B. and J. Kahn, *Mantel's theorem for random graphs* (2012), arXiv:1206.1016.

[7] Erdős, P. and M. Simonovits, *On a valence problem in extremal graph theory*, Discrete Math. **5** (1973), pp. 323–334.

[8] Kohayakawa, Y., *Szemerédi's regularity lemma for sparse graphs*, in: *Foundations of computational mathematics (Rio de Janeiro, 1997)*, Springer, Berlin, 1997 pp. 216–230.

[9] Kohayakawa, Y., T. Łuczak and V. Rödl, *On K^4-free subgraphs of random graphs*, Combinatorica **17** (1997), pp. 173–213.

[10] Mantel, W., *Problem 28*, Wiskundige Opgaven **10** (1907), pp. 60–61, (solution by H. Gouwentak, W. Mantel, J. Texeira de Mattes, F. Schuh and W.A. Wythoff).

[11] Thomassen, C., *On the chromatic number of triangle-free graphs of large minimum degree*, Combinatorica **22** (2002), pp. 591–596.