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Bounded face-width forces K_7-minors in orientable surfaces

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Abstract

Let G be a graph embedded in a nonspherical orientable surface with face-width ≥ 19. We prove that G contains a minor isomorphic to K_7.

Keywords: graph minors, graph embeddings, orientable surfaces, face-width

1 Introduction and results

A surface Σ is a compact connected 2-dimensional manifold without boundary. In the analysis boundary components, which we call *cuffs*, may emerge. We shall consider only orientable surfaces.

A simple closed curve (s.c.c.) γ in a closed surface Σ is *contractible* if it is homotopic to a constant map, and is *essential* otherwise. A s.c.c. γ is *(surface) separating* if cutting Σ along γ disconnects the surface. Every contractible s.c.c. is a separating curve, or equivalently, every nonseparating s.c.c. is essential. A *collar* of a *s.c.c.* γ is its neighbourhood which is homeomorpic

[1] Supported in part by the Research Grant P1-0297 of ARRS.

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to a cylinder (as Σ is orientable). Hence γ separates its collar into two components, which we call the *right* and *left side* of γ. We shall implicitly rely on the following fact. If a pair of s.c.c. γ, γ' intersect in a single transversal crossing, then both γ and γ' are nonseparating, and also, as Σ is orientable, nonhomotopic.

We shall consider an embedding of a graph G in Σ as a subset of Σ. The *face-width* of G, $\mathrm{fw}(G)$, is the greatest integer k, so that every essential curve $\gamma \subseteq \Sigma$ intersects G in at least k points — which we may also assume to be vertices of G. Note that the sphere contains no essential curves. When talking about face-width of a graph we will implicitly assume that it is embedded in a nonspherical surface.

The integral part of Robertson and Seymour's theory on graph minors relies on embeddings of graphs in surfaces, and face-width of a graph $G \subseteq \Sigma$ is a measure on how well G approximates the surface Σ. They have shown that if H embeds in Σ, then there is a constant c_Σ (highly dependent on the surface Σ), so that every embedding of G in Σ whose face-width exceeds c_Σ contains H as a minor [6].

Given H, can an absolute bound work for every surface of sufficiently high genus (admitting an embedding of H)? It is not difficult to show that $\mathrm{fw}(G) \geq 4$ forces a K_5-minor in G. By a result of Robertson and Vitray [7] planar graphs cannot embed in nonspherical surfaces with face-width ≥ 3, which makes G nonplanar and also non-apex. And then we apply Wagner's excluded minor theorem for K_5 [9]. Krakovski and Mohar [3] have recently shown that every graph G embedded with $\mathrm{fw}(G) \geq 6$ in an arbitrary closed surface Σ contains a K_6 minor.

Our main result states that there is a universal constant which works for every *orientable* surface in the case of K_7.

Theorem 1.1 *Let G be a graph embedded in a nonspherical orientable surface Σ with $\mathrm{fw}(G) \geq 19$. Then G contains a K_7-minor.*

We use standard graph terminology from [2] and rely on [4] for the basics on graph embeddings. The next section gives a sketch of the proof of Theorem 1.1 and we close with a handful of final remarks.

2 Execution

Fix an orientable surface Σ and an embedded graph G satisfying $\mathrm{fw}(G) \geq 19$. By a result of Robertson and Vitray [7] (see also [4, Corollary 5.5.14]) we may assume that G is 3-connected.

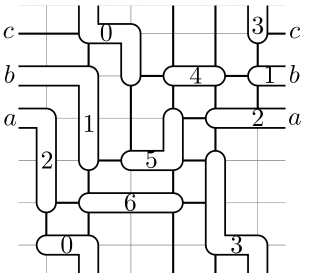

Fig. 1. K_7 minor in the toroidal 6×6 grid, we use standard identifications.

Figure 1 shows K_7 as a minor in the toroidal 6×6 grid, with ovalized shapes indicating the vertices obtained after contraction. Note that, opting for a lighter figure, we allow longer paths between ovals, and also that we only need three consecutive edges, denoted by a, b, c, which cross the left-right identification. This figure serves as a signpost for our argument. We shall construct a sequence of 6 disjoint *vertical* cycles, 6 disjoint *horizontal* paths, and an additional *triple* of paths that contract to edges denoted by a, b, c.

A *face chain* in an embedded graph is an alternating sequence

$$u_0 h_0 u_1 \dots u_{n-1} h_{n-1} u_n \tag{1}$$

of vertices $u_0, \dots, u_n$ and faces $h_0, \dots, h_{n-1}$, where every face h_i, $i = 0, \dots, n-1$, contains both u_i and u_{i+1}. The length of a face chain is equal to the number of its faces (counted with multiplicities). We call u_0 and u_n the *endvertices* of a face chain, and say a face chain is *closed* if its endvertices coincide.

Facial distance between vertices x, y is defined as the length of a shortest face chain with endvertices x and y, and vertices lying on the same face are called *cofacial*.

Now (provided the face-width is at least 2, which we assume) a closed face chain $\mathcal{F}$ without repeated vertices or faces induces a s.c.c. γ_F obtained by concatenating simple arcs between consecutive vertices in the face chain (where every such arc is contained in the appropriate face). By abusing the notation we shall call a face chain $\mathcal{F}$, for example, essential or nonseparating if $\gamma_{\mathcal{F}}$ is an essential or nonseparating curve, respectively.

Let

$$\mathcal{F}_0 = u_0 h_0 u_1 \dots u_{k-1} h_{k-1} u_0 \tag{2}$$

be *a* shortest nonseparating face chain in G and γ_0 the simple closed curve induced by $\mathcal{F}_0$. Recall the length of $\mathcal{F}_0$ satisfies $k \geq 19$.

Having fixed $\mathcal{F}_0$ and γ_0 we can state the following proposition.

Proposition 2.1 *Let $\mathcal{F}'$ be a face chain of length ≤ 9 with endvertices u_i and u_j, $i, j \in \{0, \ldots, k-1\}$. Then every essential curve in $\bigcup \mathcal{F}_0 \cup \bigcup \mathcal{F}'$ is homotopic to γ_0.*

By induction on the length of $\mathcal{F}'$ we may assume that $\mathcal{F}'$ induces a simple curve γ', so that $\gamma' \cap \gamma_0 = \{u_i, u_j\}$. The vertices u_i, u_j split γ_0 into a pair of arcs γ_0^{ij} and γ_0^{ji} and the combined length of closed curves $\gamma_0^{ij} \cup \gamma'$ and $\gamma_0^{ji} \cup \gamma'$ is at most $k+18$. By the three paths property [8] (see also [4, Proposition 4.3.1]) it is not possible that both $\gamma_0^{ij} \cup \gamma'$, $\gamma_0^{ji} \cup \gamma'$ are separating, and consequently one of $\gamma_0^{ij} \cup \gamma'$, $\gamma_0^{ji} \cup \gamma'$ is contractible and the other is homotopic to γ_0.

As $\mathrm{fw}(G) \geq 19$ a result of Brunet, Mohar, and Richter [1, Theorem 6.1(1)] implies:

Proposition 2.2 *There exists a collection of nested disjoint homotopic cycles $\mathcal{C} = \{C_0, \ldots, C_7\}$ so that (i) C_4 is contained in $\bigcup \mathcal{F}_0$, (ii) all cycles in $\mathcal{C}$ are homotopic to γ_0, (iii) no pair of vertices x, y, $x \in C_0, y \in C_7$, are cofacial, and (iv) every vertex $x \in C_0 \cup C_7$ is at facial distance at most 4 to some vertex in $\{u_0, \ldots, u_{k-1}\}$.*

For convenience reasons let us fix an orientation of C_0 and extend the orientation to C_7 by homotopy. We shall later possibly perturb the intermediate cycles $C_1, \ldots, C_6$.

Let Δ denote the cylinder bordered by C_0 and C_7 (which contains cycles $C_1, \ldots, C_6$), and let $\overline{\Delta}$ denote the complementary surface. Let $G[\Delta]$ be the subgraph of G induced by Δ, and let $G[\overline{\Delta}]$ be defined analogously. Proposition 2.2(iv) implies that Δ is minimal possible. If x is an edge or a vertex in $C_0 \cup C_7$ then the graph $G[\Delta] - x$ does not contain eight disjoint cycles homotopic to C_0.

Proposition 2.3 *There exists $C_0 - C_7$ paths*[3] *Q_a, Q_b, Q_c in $G[\overline{\Delta}]$, so that the cyclic orderings of their respective endvertices a_0, b_0, c_0 along C_0 and a_7, b_7, c_7 along C_7 match the orientations of both C_0 and C_7.*

The reasoning goes as follows. Let

$$\mathcal{F}_1 = x_0 g_0 x_1 \ldots x_{\ell-1} g_{\ell-1} x_\ell \tag{3}$$

be *a* shortest face chain lying in $\overline{\Delta}$ whose endvertices x_0 and x_ℓ lie on C_7 and C_0, respectively. By Proposition 2.2(iii) we have $\ell \geq 2$.

[3] For an $A - B$ path we will require/assume that none of its internal vertices lie in $A \cup B$.

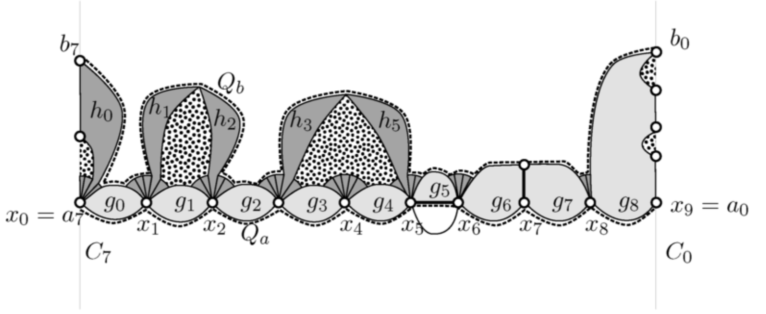

Fig. 2. Construction of paths Q_a and Q_b.

Now $\mathcal{F}_1$ induces a simple curve γ_1, and we may assume that its left side intersects C_0 in the *forward* direction. A face $f \subseteq \overline{\Delta}$ incident with x_i is a *left face* if f does not intersect the right side of γ_1. We may also assume that the number of left faces is as small as possible. This implies, for example, that no left face incident with x_1 intersects C_7.

We concatenate *right* segments of faces of $\mathcal{F}_1$ to form a $C_7 - C_0$ path Q_a. The path Q_b is the boundary of the union of all faces that lie immediately to the left of Q_a — see Figure 2 where the left side of Q_a is depicted above Q_a.

Let a_7, b_7 be the endvertices of Q_a, Q_b in C_7, and, similarly, let a_0, b_0 be their respective endvertices in C_0. We may also assume that $x_0 = a_7$ and $x_\ell = a_0$.

Note that by construction

$$\begin{array}{l}\text{every vertex } v \in Q_a \cup Q_b \text{ is cofacial with (at least) one of } x_0, \ldots, x_\ell, \\ \text{and also that } a_7, b_7 \text{ and } a_0, b_0 \text{ are pairs of cofacial vertices.}\end{array} \tag{4}$$

Observe that the dotted regions, depicted in Figure 2, are disks. For example, the closed face chain containing faces g_3, g_4, h_5, h_3 is too short to be essential. On the other hand, every vertex of $g_8 \cap C_0$ is at facial distance ≤ 4 to some vertex of $\mathcal{F}_0$ by Proposition 2.2(iv). Now Proposition 2.1(iii) implies that the dotted regions between g_8 and C_0 are contained in a disk bounded by g_8 and C_0 (similar observation holds for dotted regions between h_0 and C_7).

We denote by Δ_0 the disk bordered by Q_a, Q_b and segments of C_0 and C_7 which contains $\bigcup \mathcal{F}_0$. Let S_0 and S_7 be maximal subpaths of C_0 and C_7, respectively, which are disjoint with Δ_0.

The only possible obstruction to a $S_7 - S_0$ path in $\overline{\Delta} \setminus (Q_a \cup Q_b)$ is a face f in $\overline{\Delta} \setminus \Delta_0$ containing a pair of vertices s_a, s_b, so that $s_a \in Q_a$ and $s_b \in Q_b$. By (4) there exists indices i, j, so that s_a is cofacial with x_i and s_b is cofacial with x_j. Now $|i - j| \geq 4$ contradicts the fact that $\mathcal{F}_1$ was chosen as short as possible.

On the other hand $|i-j| \leq 3$ implies that there exists a nonseparating face chain of length ≤ 6 (as it would intersect the curve γ_1 in a single transversal crossing). This is also absurd.

Hence there exists a $S_7 - S_0$ path in $\Delta \setminus (Q_a \cup Q_b)$, which we denote by Q_c. As Δ_0 is a disk the orderings of endvertices of paths Q_a, Q_b, Q_c satisfy Proposition 2.3.

Minimal length of $\mathcal{F}_0$ and the planar version of Menger theorem (see for example [5]) imply:

Proposition 2.4 *There exists a family of k disjoint $C_0 - C_7$ paths $\mathcal{P} = \{P_0, \ldots, P_{k-1}\}$ in $G[\Delta]$.*

The paths' indices are taken modulo k, and we may assume that for every i the endvertex of P_{i+1} along C_7, denoted by p_{i+1}, immediately follows the one of P_i in the orientation of C_7. As Δ is a cylinder their respective endvertices along C_0, which we denote by p'_i, also match the orientation.

Let us choose a collection of k paths $\mathcal{P}$ so that the numbers of their endvertices in $C_7 \cap \Delta_0$ and also in $C_0 \cap \Delta_0$ are as close to $k-1$ as possible. Note that we can optimize these numbers independently, as $\mathcal{F}_0$ induces a minimal C_0, C_7 separator in Δ whose order k is the same as $|\mathcal{P}|$.

Now let us show that it is not possible that, say, all of $p_0, \ldots, p_{k-1}$ belong to $C_7 \cap \Delta_0$. As a_7 and b_7 are cofacial, there exists a vertex $s \in S_7$ which is incident with an edge $e \in G[\Delta] - E(C_7)$. Let K be the $(C_7 \cup \mathcal{P})$-bridge containing e. As Δ is minimal K has no attachment vertices in $C_7 - s$. And as G is 3-connected K attaches to one of the paths in $\mathcal{P}$, which we can then reroute.

Let us assume that

$$\text{at least two of the paths in } \mathcal{P} \text{ attach to } C_0 \cap \Delta_0 \text{ and at least two paths in } \mathcal{P} \text{ attach to } C_7 \cap \Delta. \tag{5}$$

In this case we have.

Proposition 2.5 *There exists a linkage between Q_a, Q_b, Q_c and $p_{i_0}, p_{i_0+1}, p_{i_0+2}$ in C_7 and also a linkage between Q_a, Q_b, Q_c and $p'_{j_0}, p'_{j_0+1}, p'_{j_0+2}$ in C_0, for some choice of $i_0, j_0 \in \{0, \ldots, k-1\}$.*

We can, say, for the latter choose j_0+1 as the last index so that P_{j_0+1} attaches to $C_0 \cap \Delta_0$.

A *crossing* of a path P and a cycle C is a connected component of their intersection.

Proposition 2.6 *We can perturb $\mathcal{C}$ and $\mathcal{P}$ (given by Propositions 2.2 and 2.4) so that (i) every path in $\mathcal{P}$ and every cycle in $\mathcal{C}$ cross exactly once, while (ii) keeping C_0 and C_7 fixed and also (iii) not altering the endvertices of paths in $\mathcal{P}$.*

This can be done by minimizing the total number of edges contained in the union of $\mathcal{P}$ and $\mathcal{C}$. It is not difficult to argue that no path from $\mathcal{P}$ has two consecutive crossings with the same cycle from $\mathcal{C}$, and vice versa. So let us choose a cycle C_j, with j as small as possible, which crosses some path from $\mathcal{P}$ twice. By construction, $j \neq 0$. Let us also choose two crossings of C_j and some path P_i, so that the intermediate segment $P \subseteq P_i$ does not cross C_{j-1} (it has to cross C_{j+1}, though) and is as short as possible. Now some $P \in \{P_{i-1}, P_{i+1}\}$ crosses the disk bounded by P and C_j, and consequently crosses C_{j-1} twice, which is absurd.

Finally let $\mathcal{C}_6 = \{C_1, \ldots, C_6\}$ be the family of 6 cycles. Assuming that $i_0 = 0$ and $j_0 \leq k/2$ (we can do that by renumbering of paths in $\mathcal{P}$) let us define a family $C_0 - C_7$ paths $\mathcal{P}_6 = \{P'_1, \ldots, P'_6\}$ by the following construction. For $i = 1, 2, 3$, let $P'_{7-i} = P_{k-i}$ and let P'_i be obtained by concatenating the initial segment of P_{i-1}, a segment of C_{4-i} in the middle, and the terminal segment of P_{j_0-1+i}.

The linkages according to Proposition 2.5 and paths Q_a, Q_b, Q_c (by Proposition 2.3) give rise to a triple of cycles crossing all cycles in $\mathcal{C}_6$. Together with $\mathcal{C}_6$ and $\mathcal{P}_6$ we have a K_7-minor by Figure 1.

We shall skip the sketch in case the original choice of $\mathcal{P}$ cannot satisfy (5). In such a case the segments S_0 and S_7, where the path Q_c can attach to, have to be chosen with additional care in order to allow an appropriate linkage in Proposition 2.5. The obstruction to the existence of Q_c in this case is again a single face f, yet the of vertices s_a, a_b can lie in $(C_0 \cup C_7) \setminus (Q_a \cup Q_b)$, and the arguments needed to eliminate all possible obstructions are more elaborate.

3 Concluding remarks

Most likely the constant 19 from Theorem 1.1 is not optimal. In order to keep the presentation as clear as possible we have focused on a single realization of K_7 as a minor in the toroidal 6×6 grid. Allowing several types of K_7 models might allow us to reduce the face-width bound at the expense of making the proof not as focused and much longer.

A direct generalization of Theorem 1.1 to nonorientable surfaces would have to exclude both projective plane and Klein bottle, as K_7 embeds in neither of them. Now N_3, the nonorientable surface of genus 3, which could take torus' place in our proof has several drawbacks. On one hand, it contains no *generic grid* and also admits a number of nonequivalent embeddings of K_7. Finally our approach relied on having no orientation reversing s.c.c., their inclusion would make the analysis much more difficult.

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