Bounded face-width forces $K_7$-minors in orientable surfaces

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Abstract

Let $G$ be a graph embedded in a nonspherical orientable surface with face-width $\geq 19$. We prove that $G$ contains a minor isomorphic to $K_7$.

\textit{Keywords:} graph minors, graph embeddings, orientable surfaces, face-width

1 Introduction and results

A surface $\Sigma$ is a compact connected 2-dimensional manifold without boundary. In the analysis boundary components, which we call cuffs, may emerge. We shall consider only orientable surfaces.

A simple closed curve (s.c.c.) $\gamma$ in a closed surface $\Sigma$ is contractible if it is homotopic to a constant map, and is essential otherwise. A s.c.c. $\gamma$ is (surface) separating if cutting $\Sigma$ along $\gamma$ disconnects the surface. Every contractible s.c.c. is a separating curve, or equivalently, every nonseparating s.c.c. is essential. A collar of a s.c.c. $\gamma$ is its neighbourhood which is homeomorphic

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to a cylinder (as $\Sigma$ is orientable). Hence $\gamma$ separates its collar into two components, which we call the right and left side of $\gamma$. We shall implicitly rely on the following fact. If a pair of s.c.c. $\gamma, \gamma'$ intersect in a single transversal crossing, then both $\gamma$ and $\gamma'$ are nonseparating, and also, as $\Sigma$ is orientable, nonhomotopic.

We shall consider an embedding of a graph $G$ in $\Sigma$ as a subset of $\Sigma$. The face-width of $G$, $\text{fw}(G)$, is the greatest integer $k$, so that every essential curve $\gamma \subseteq \Sigma$ intersects $G$ in at least $k$ points — which we may also assume to be vertices of $G$. Note that the sphere contains no essential curves. When talking about face-width of a graph we will implicitly assume that it is embedded in a nonspherical surface.

The integral part of Robertson and Seymour’s theory on graph minors relies on embeddings of graphs in surfaces, and face-width of a graph $G \subseteq \Sigma$ is a measure on how well $G$ approximates the surface $\Sigma$. They have shown that if $H$ embeds in $\Sigma$, then there is a constant $c_\Sigma$ (highly dependent on the surface $\Sigma$), so that every embedding of $G$ in $\Sigma$ whose face-width exceeds $c_\Sigma$ contains $H$ as a minor [6].

Given $H$, can an absolute bound work for every surface of sufficiently high genus (admitting an embedding of $H$)? It is not difficult to show that $\text{fw}(G) \geq 4$ forces a $K_5$-minor in $G$. By a result of Robertson and Vitray [7] planar graphs cannot embed in nonspherical surfaces with face-width $\geq 3$, which makes $G$ nonplanar and also non-apex. And then we apply Wagner’s excluded minor theorem for $K_5$ [9]. Krakovski and Mohar [3] have recently shown that every graph $G$ embedded with $\text{fw}(G) \geq 6$ in an arbitrary closed surface $\Sigma$ contains a $K_6$ minor.

Our main result states that there is a universal constant which works for every orientable surface in the case of $K_7$.

**Theorem 1.1** Let $G$ be a graph embedded in a nonspherical orientable surface $\Sigma$ with $\text{fw}(G) \geq 19$. Then $G$ contains a $K_7$-minor.

We use standard graph terminology from [2] and rely on [4] for the basics on graph embeddings. The next section gives a sketch of the proof of Theorem 1.1 and we close with a handful of final remarks.

## 2 Execution

Fix an orientable surface $\Sigma$ and an embedded graph $G$ satisfying $\text{fw}(G) \geq 19$. By a result of Robertson and Vitray [7] (see also [4, Corollary 5.5.14]) we may assume that $G$ is 3-connected.
Fig. 1. $K_7$ minor in the toroidal $6 \times 6$ grid, we use standard identifications.

Figure 1 shows $K_7$ as a minor in the toroidal $6 \times 6$ grid, with ovalized shapes indicating the vertices obtained after contraction. Note that, opting for a lighter figure, we allow longer paths between ovals, and also that we only need three consecutive edges, denoted by $a, b, c$, which cross the left-right identification. This figure serves as a signpost for our argument. We shall construct a sequence of 6 disjoint vertical cycles, 6 disjoint horizontal paths, and an additional triple of paths that contract to edges denoted by $a, b, c$.

A face chain in an embedded graph is an alternating sequence

$$u_0 h_0 u_1 ... u_{n-1} h_{n-1} u_n$$

of vertices $u_0, ..., u_n$ and faces $h_0, ..., h_{n-1}$, where every face $h_i, i = 0, ..., n - 1$, contains both $u_i$ and $u_{i+1}$. The length of a face chain is equal to the number of its faces (counted with multiplicities). We call $u_0$ and $u_n$ the endvertices of a face chain, and say a face chain is closed if its endvertices coincide.

Facial distance between vertices $x, y$ is defined as the length of a shortest face chain with endvertices $x$ and $y$, and vertices lying on the same face are called cofacial.

Now (provided the face-width is at least 2, which we assume) a closed face chain $F$ without repeated vertices or faces induces a s.c.c. $\gamma_F$ obtained by concatenating simple arcs between consecutive vertices in the face chain (where every such arc is contained in the appropriate face). By abusing the notation we shall call a face chain $F$, for example, essential or nonseparating if $\gamma_F$ is an essential or nonseparating curve, respectively.

Let

$$F_0 = u_0 h_0 u_1 ... u_{k-1} h_{k-1} u_0$$

be a shortest nonseparating face chain in $G$ and $\gamma_0$ the simple closed curve induced by $F_0$. Recall the length of $F_0$ satisfies $k \geq 19$. 
Having fixed $\mathcal{F}_0$ and $\gamma_0$ we can state the following proposition.

**Proposition 2.1** Let $\mathcal{F}'$ be a face chain of length $\leq 9$ with endvertices $u_i$ and $u_j$, $i, j \in \{0, \ldots, k - 1\}$. Then every essential curve in $\bigcup \mathcal{F}_0 \cup \bigcup \mathcal{F}'$ is homotopic to $\gamma_0$.

By induction on the length of $\mathcal{F}'$ we may assume that $\mathcal{F}'$ induces a simple curve $\gamma'$, so that $\gamma' \cap \gamma_0 = \{u_i, u_j\}$. The vertices $u_i, u_j$ split $\gamma_0$ into a pair of arcs $\gamma_{ij}^0$ and $\gamma_{ji}^0$ and the combined length of closed curves $\gamma_{ij}^0 \cup \gamma'$ and $\gamma_{ji}^0 \cup \gamma'$ is at most $k + 18$. By the three paths property [8] (see also [4, Proposition 4.3.1]) it is not possible that both $\gamma_{ij}^0 \cup \gamma'$, $\gamma_{ji}^0 \cup \gamma'$ are separating, and consequently one of $\gamma_{ij}^0 \cup \gamma'$, $\gamma_{ji}^0 \cup \gamma'$ is contractible and the other is homotopic to $\gamma_0$.

As $\text{fw}(G) \geq 19$ a result of Brunet, Mohar, and Richter [1, Theorem 6.1(1)] implies:

**Proposition 2.2** There exists a collection of nested disjoint homotopic cycles $\mathcal{C} = \{C_0, \ldots, C_7\}$ so that (i) $C_4$ is contained in $\bigcup \mathcal{F}_0$, (ii) all cycles in $\mathcal{C}$ are homotopic to $\gamma_0$, (iii) no pair of vertices $x, y$, $x \in C_0, y \in C_7$, are cofacial, and (iv) every vertex $x \in C_0 \cup C_7$ is at facial distance at most 4 to some vertex in $\{u_0, \ldots, u_{k - 1}\}$.

For convenience reasons let us fix an orientation of $C_0$ and extend the orientation to $C_7$ by homotopy. We shall later possibly perturb the intermediate cycles $C_1, \ldots, C_6$.

Let $\Delta$ denote the cylinder bordered by $C_0$ and $C_7$ (which contains cycles $C_1, \ldots, C_6$), and let $\overline{\Delta}$ denote the complementary surface. Let $G[\Delta]$ be the subgraph of $G$ induced by $\Delta$, and let $G[\overline{\Delta}]$ be defined analogously. Proposition 2.2(iv) implies that $\Delta$ is minimal possible. If $x$ is an edge or a vertex in $C_0 \cup C_7$ then the graph $G[\Delta] - x$ does not contain eight disjoint cycles homotopic to $C_0$.

**Proposition 2.3** There exists $C_0 - C_7$ paths $Q_a, Q_b, Q_c$ in $G[\overline{\Delta}]$, so that the cyclic orderings of their respective endvertices $a_0, b_0, c_0$ along $C_0$ and $a_7, b_7, c_7$ along $C_7$ match the orientations of both $C_0$ and $C_7$.

The reasoning goes as follows. Let

$$\mathcal{F}_1 = x_0 g_0 x_1 \ldots x_{\ell - 1} g_{\ell - 1} x_\ell$$

be a shortest face chain lying in $\overline{\Delta}$ whose endvertices $x_0$ and $x_\ell$ lie on $C_7$ and $C_0$, respectively. By Proposition 2.2(iii) we have $\ell \geq 2$.

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3 For an $A - B$ path we will require/assume that none of its internal vertices lie in $A \cup B$. 

Now $\mathcal{F}_1$ induces a simple curve $\gamma_1$, and we may assume that its left side intersects $C_0$ in the forward direction. A face $f \subseteq \Delta$ incident with $x_i$ is a left face if $f$ does not intersect the right side of $\gamma_1$. We may also assume that the number of left faces is as small as possible. This implies, for example, that no left face incident with $x_1$ intersects $C_7$.

We concatenate right segments of faces of $\mathcal{F}_1$ to form a $C_7 - C_0$ path $Q_a$. The path $Q_b$ is the boundary of the union of all faces that lie immediately to the left of $Q_a$ — see Figure 2 where the left side of $Q_a$ is depicted above $Q_a$.

Let $a_7, b_7$ be the endvertices of $Q_a$, $Q_b$ in $C_7$, and, similarly, let $a_0, b_0$ be their respective endvertices in $C_0$. We may also assume that $x_0 = a_7$ and $x_\ell = a_0$.

Note that by construction

\begin{equation}
\text{every vertex } v \in Q_a \cup Q_b \text{ is cofacial with (at least) one of } x_0, \ldots, x_\ell, \text{ and also that } a_7, b_7 \text{ and } a_0, b_0 \text{ are pairs of cofacial vertices.}
\end{equation}

Observe that the dotted regions, depicted in Figure 2, are disks. For example, the closed face chain containing faces $g_3, g_4, h_5, h_3$ is too short to be essential. On the other hand, every vertex of $g_8 \cap C_0$ is at facial distance $\leq 4$ to some vertex of $\mathcal{F}_0$ by Proposition 2.2(iv). Now Proposition 2.1(iii) implies that the dotted regions between $g_8$ and $C_0$ are contained in a disk bounded by $g_8$ and $C_0$ (similar observation holds for dotted regions between $h_0$ and $C_7$).

We denote by $\Delta_0$ the disk bordered by $Q_a, Q_b$ and segments of $C_0$ and $C_7$ which contains $\bigcup \mathcal{F}_0$. Let $S_0$ and $S_7$ be maximal subpaths of $C_0$ and $C_7$, respectively, which are disjoint with $\Delta_0$.

The only possible obstruction to a $S_7 - S_0$ path in $\overline{\Delta} \setminus (Q_a \cup Q_b)$ is a face $f$ in $\overline{\Delta} \setminus \Delta_0$ containing a pair of vertices $s_a, s_b$, so that $s_a \in Q_a$ and $s_b \in Q_b$. By (4) there exists indices $i, j$, so that $s_a$ is cofacial with $x_i$ and $s_b$ is cofacial with $x_j$. Now $|i - j| \geq 4$ contradicts the fact that $\mathcal{F}_1$ was chosen as short as possible.
On the other hand $|i - j| \leq 3$ implies that there exists a nonseparating face chain of length $\leq 6$ (as it would intersect the curve $\gamma_1$ in a single transversal crossing). This is also absurd.

Hence there exists a $S_7 - S_0$ path in $\Delta \setminus (Q_a \cup Q_b)$, which we denote by $Q_c$. As $\Delta_0$ is a disk the orderings of endvertices of paths $Q_a, Q_b, Q_c$ satisfy Proposition 2.3.

Minimal length of $F_0$ and the planar version of Menger theorem (see for example [5]) imply:

**Proposition 2.4** There exists a family of $k$ disjoint $C_0 - C_7$ paths $\mathcal{P} = \{P_0, \ldots, P_{k-1}\}$ in $G[\Delta]$.

The paths’ indices are taken modulo $k$, and we may assume that for every $i$ the endvertex of $P_{i+1}$ along $C_7$, denoted by $p_{i+1}$, immediately follows the one of $P_i$ in the orientation of $C_7$. As $\Delta$ is a cylinder their respective endvertices along $C_0$, which we denote by $p'_i$, also match the orientation.

Let us choose a collection of $k$ paths $\mathcal{P}$ so that the numbers of their endvertices in $C_7 \cap \Delta_0$ and also in $C_0 \cap \Delta_0$ are as close to $k - 1$ as possible. Note that we can optimize these numbers independently, as $F_0$ induces a minimal $C_0, C_7$ separator in $\Delta$ whose order $k$ is the same as $|\mathcal{P}|$.

Now let us show that it is not possible that, say, all of $p_0, \ldots, p_{k-1}$ belong to $C_7 \cap \Delta_0$. As $a_7$ and $b_7$ are cofacial, there exists a vertex $s \in S_7$ which is incident with an edge $e \in G[\Delta] - E(C_7)$. Let $K$ be the $(C_7 \cup \mathcal{P})$-bridge containing $e$. As $\Delta$ is minimal $K$ has no attachment vertices in $C_7 - s$. And as $G$ is 3-connected $K$ attaches to one of the paths in $\mathcal{P}$, which we can then reroute.

Let us assume that

at least two of the paths in $\mathcal{P}$ attach to $C_0 \cap \Delta_0$ and at least two paths in $\mathcal{P}$ attach to $C_7 \cap \Delta$. (5)

In this case we have.

**Proposition 2.5** There exists a linkage between $Q_a, Q_b, Q_c$ and $p_{i_0}, p_{i_0+1}, p_{i_0+2}$ in $C_7$ and also a linkage between $Q_a, Q_b, Q_c$ and $p'_{j_0}, p'_{j_0+1}, p'_{j_0+2}$ in $C_0$, for some choice of $i_0, j_0 \in \{0, \ldots, k-1\}$.

We can, say, for the latter choose $j_0 + 1$ as the last index so that $P_{j_0+1}$ attaches to $C_0 \cap \Delta_0$.

A crossing of a path $P$ and a cycle $C$ is a connected component of their intersection.
Proposition 2.6 We can perturb $C$ and $P$ (given by Propositions 2.2 and 2.4) so that (i) every path in $P$ and every cycle in $C$ cross exactly once, while (ii) keeping $C_0$ and $C_7$ fixed and also (iii) not altering the endvertices of paths in $P$.

This can be done by minimizing the total number of edges contained in the union of $P$ and $C$. It is not difficult to argue that no path from $P$ has two consecutive crossings with the same cycle from $C$, and vice versa. So let us choose a cycle $C_j$, with $j$ as small as possible, which crosses some path from $P$ twice. By construction, $j \neq 0$. Let us also choose two crossings of $C_j$ and some path $P_i$, so that the intermediate segment $P \subseteq P_i$ does not cross $C_{j-1}$ (it has to cross $C_{j+1}$, though) and is as short as possible. Now some $P \in \{P_{i-1}, P_{i+1}\}$ crosses the disk bounded by $P$ and $C_j$, and consequently crosses $C_{j-1}$ twice, which is absurd.

Finally let $C_6 = \{C_1, \ldots, C_6\}$ be the family of 6 cycles. Assuming that $i_0 = 0$ and $j_0 \leq k/2$ (we can do that by renumbering of paths in $P$) let us define a family $C_0 - C_7$ paths $P_6 = \{P'_1, \ldots, P'_6\}$ by the following construction. For $i = 1, 2, 3$, let $P'_i = P'_{k-i}$ and let $P'_i$ be obtained by concatenating the initial segment of $P_{i-1}$, a segment of $C_{4-i}$ in the middle, and the terminal segment of $P_{j_0-1+i}$.

The linkages according to Proposition 2.5 and paths $Q_a, Q_b, Q_c$ (by Proposition 2.3) give rise to a triple of cycles crossing all cycles in $C_6$. Together with $C_6$ and $P_6$ we have a $K_7$-minor by Figure 1.

We shall skip the sketch in case the original choice of $P$ cannot satisfy (5). In such a case the segments $S_0$ and $S_7$, where the path $Q_c$ can attach to, have to be chosen with additional care in order to allow an appropriate linkage in Proposition 2.5. The obstruction to the existence of $Q_c$ in this case is again a single face $f$, yet the of vertices $s_a, a_b$ can lie in $(C_0 \cup C_7) \setminus (Q_a \cup Q_b)$, and the arguments needed to eliminate all possible obstructions are more elaborate.

3 Concluding remarks

Most likely the constant 19 from Theorem 1.1 is not optimal. In order to keep the presentation as clear as possible we have focused on a single realization of $K_7$ as a minor in the toroidal $6 \times 6$ grid. Allowing several types of $K_7$ models might allow us to reduce the face-width bound at the expense of making the proof not as focused and much longer.

A direct generalization of Theorem 1.1 to nonorientable surfaces would have to exclude both projective plane and Klein bottle, as $K_7$ embeds in
neither of them. Now $N_3$, the nonorientable surface of genus 3, which could take torus’ place in our proof has several drawbacks. On one hand, it contains no generic grid and also admits a number of nonequivalent embeddings of $K_7$. Finally our approach relied on having no orientation reversing s.c.c., their inclusion would make the analysis much more difficult.

References


