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Colorings of hypergraphs with large number of colors

Ilia Akolzin [1,2]

Faculty of Innovations and High Technology, Department of Discrete Mathematics
Moscow Institute of Physics and Technology
Moscow, Russian Federation

Dmitry Shabanov [1,3]

Faculty of Mechanics and Mathematics, Department of Probability Theory
Lomonosov Moscow State University
Moscow, Russian Federation

Abstract

The paper deals with the well-known problem of Erdős and Hajnal concerning colorings of uniform hypergraphs and some related questions. Let $m(n,r)$ denote the minimum possible number of edges in an n-uniform non-r-colorable hypergraph. We show that for $r > n$,

$$c_1 \frac{n}{\ln n} \leqslant \frac{m(n,r)}{r^n} \leqslant C_1 n^3 \ln n,$$

where $c_1, C_1 > 0$ are some absolute constants.

Keywords: colorings of hypergraphs, property B problem, Turán number.

1 Introduction

The paper deals the classical extremal combinatorial problem of P. Erdős and A. Hajnal concerning colorings of hypergraphs. Let us recall some definitions.

A *vertex coloring* of a hypergraph $H = (V, E)$ is a mapping $f : V \to \mathbb{N}$. Coloring f is said to be *proper* for H if there is no monochromatic edges in this coloring. A hypergraph is called r*-colorable* if there is a proper coloring with r colors for it. *The chromatic number* of the hypergraph H, $\chi(H)$, is the least r such that H is r colorable, i.e. the minimum number of colors required for a proper coloring of H.

In 1961 P. Erdős and A. Hajnal proposed (see [1]) to determine the value $m(n, r)$ equal to the minimum possible number of edges in an n-uniform non-r-colorable hypergraph. Formally,

$$m(n, r) = \min\{|E| : H = (V, E) \text{ is } n\text{-uniform}, \ \chi(H) > r\}.$$

This problem, especially its 2-coloring case (Property B problem), has played a significant role in the development of probabilistic methods in combinatorics.

In 1963-64 Erdős showed that for hypergraphs the behavior of $m(n, 2)$ is quite different. Similar estimates for $m(n, r)$ obtained by the same way are the following (e.g., see [2]):

$$r^{n-1} \leqslant m(n, r) \leqslant \frac{e}{2} n^2 r^n \ln r \left(1 + O\left(\frac{1}{n}\right)\right). \tag{1}$$

The improvement of the lower bound in (1) has a long history (the reader is referred to the survey [2] for the details). The best current estimates for constant number of colors and large uniformity parameter were obtained by D. Cherkashin and J. Kozik (see [3]) for $r > 2$,

$$m(n, r) = \Omega\left(\left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}} r^{n-1}\right). \tag{2}$$

In case when $r = 2$ the best result was obtained by J. Radhakrishnan and A. Srinivasan (see [4])

In the current paper we study the Erdős–Hajnal problem in the case when r is large and n small. In [5] N. Alon showed that for $r \gg n$, the estimate (1)

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[2] Email: iakolzin@gmail.com
[3] Email: dm.shabanov.msu@gmail.com

is far away from the right answer. He proved that if $n \to \infty$ and $r/n \to \infty$ then

$$m(n,r) = O\left(n^{5/2}(\ln n)\left(\frac{3}{4}\right)^n\binom{r(n-1)+1}{n}\right). \tag{3}$$

The first result of the paper refines Alon's result (3) as follows.

Theorem 1.1 *Suppose $r > n$. Then*

$$m(n,r) = O\left(n^{7/2}(\ln n)\left(\frac{1}{e}\right)^n\binom{r(n-1)+1}{n}\right) = O\left(n^3(\ln n)r^n\right). \tag{4}$$

The first lower bound for $m(n,r)$ of order r^n for large values of r (note that the bounds (1), (2) give only r^{n-1}) was also obtained by Alon in [5]. He showed that

$$m(n,r) > (n-1)\left\lceil\frac{r}{n}\right\rceil\left\lfloor\frac{n-1}{n}r\right\rfloor^{n-1}, \tag{5}$$

which for $r > n$, implies that $m(n,r) = \Omega(r^n)$. A better result can be established by the help of a criterion for r-colorability of an arbitrary hypergraph in terms of so-called *ordered r-chains* proved by A. Pluhár in [6] (see Section 2 for the details). By using this criterion and some additional observation concerning the number of ordered r-chains Shabanov (see [2]) showed that for $r > n$ $m(n,r) = \Omega\left(n^{1/2}r^n\right)$.

We prove the following extension of the results (2) and (5).

Theorem 1.2 *There exists an absolute constant $c_1 > 0$ such that for any $r > n$,*

$$m(n,r) \geqslant c_1\left(\frac{n}{\ln n}\right)r^n. \tag{6}$$

One can consider the following "local" generalization of the extremal value $m(n,r)$. Let $d(n,r)$ denote the minimum possible value of the maximum edge degree in an n-uniform hypergraph with chromatic number greater than r. Recall that the degree of an edge is the number of other edges of a hypergraph intersecting this edge. The maximum edge degree of a hypergraph H is denoted by $D(H)$.

In the current paper we also establish new bounds for the local variant of the problem.

Theorem 1.3 *There exist some absolute constants c, $C > 0$ such that for any $r > n \geqslant 3$,*

$$c \cdot \frac{n}{\ln n} \leqslant \frac{d(n,r)}{r^{n-1}} \leqslant C \cdot n^3 \ln n. \tag{7}$$

2 Ideas of the proofs

In the current section we give the main ideas that underlie the proofs of our results. We shall start with the lower bound in Theorem 1.2.

2.1 *Lower bound for $m(n, r)$*

The proof is just a combination of Alon's idea from [5] with the argument of Cherkashin and Kozik from [3]. Let $H = (V, E)$ be an n-uniform hypergraph with small number of edges:

$$|E| \leqslant c(n-1)b\left(\frac{n}{\ln n}\right)^{\frac{a-1}{a}} a^{n-1}, \tag{8}$$

where $a = \lfloor \frac{n-1}{n} r \rfloor$ and $b = r - a = \lceil \frac{r}{n} \rceil$. We want to show that there exists a proper coloring with r colors for H.

A remarkable approach for establishing the r-colorability of a hypergraph is the following criterion, obtained by Pluhár in [6]. Suppose $|V| = N$ and let $\sigma : V \to \{1, \ldots, N\}$ be an ordering of the vertices of H. An ordered tuple $(A_1, \ldots, A_r)$ is said to form *an ordered r-chain* with respect to σ if for every $j = 1, \ldots, r-1$, $|A_j \cap A_{j+1}| = 1$ and $\sigma(v) \leqslant \sigma(u)$ for any $v \in A_j$, $u \in A_{j+1}$. The next statement (Pluhár's criterion) connects the existence of a proper r-coloring with the presence of an ordering without ordered r-chains.

Pluhár's criterion. *A hypergraph is r-colorable if and only if there exists an ordering of its vertex set without ordered r-chains.*

Let $(X_v, v \in V)$ be a set of independent random variables with uniform distribution on $[0, 1]$. For every edge $A \in E$, we introduce $f(A) = \min_{v \in A} X_v$, $\quad l(A) = \max_{v \in A} X_v$. Let $p = \frac{2 \ln n}{n} \in (0, 1)$. An edge A is called *bad* if the following event holds

$$\mathcal{B}(A) = \left\{ l(A) - f(A) \leqslant \frac{1-p}{a} \right\}.$$

The edges, which are not bad, are called *good*. The set of the all good edges forms a random hypergraph H'. We show by using Pluhár's criterion that H' is a-colorable with positive probability. Moreover, we prove that with probability at least $1/3$ the events $X < (n-1)b$ and $\chi(H') \leqslant a$ hold simultaneously. Here X denotes the number of bad edges.

Let us fix a proper coloring of H' with a colors $\{1, 2, \ldots, a\}$. Then only bad edges can be monochromatic in the hypergraph H. We can use the remained $(r - a)$ colors to recolor some vertices from them and get a proper coloring of H with r colors.

2.2 Upper bounds for $m(n,r)$ and $d(n,r)$

Now we give a sketch of proof for the upper bounds. Alon's approach (see [5]) to estimating $m(n,r)$ from above uses a close connection between the value $m(n,r)$ and the Turán numbers. Recall that the Turán number $T(v,b,n)$ is the minimum possible number of edges in a v-vertex n-uniform hypergraph such that any of its b-vertex subsets contains an edge of the hypergraph. Alon established the following relation between $m(n,r)$ and the Turán numbers:

$$m(n,r) \leqslant \min_{b \geqslant n} T(r(b-1)+1, b, n).$$

We improve Alon's bound by using the construction, proposed by A. Sidorenko (see [7]), who gave a good upper bound for the Turán density, and also some optimization over the parameter b. Similar approach is used for the proof of the local variant, here we have to adapt Sidorenko's construction (we have to make some regularization for it) to obtain the bound for the maximum edge degree.

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