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A SAT attack on the Erdős–Szekeres conjecture

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Abstract

A classical conjecture of Erdős and Szekeres states that every set of $2^{k-2}+1$ points in the plane in general position contains k points in convex position. In 2006, Peters and Szekeres introduced the following stronger conjecture: every red-blue coloring of the edges of the ordered complete 3-uniform hypergraph on $2^{k-2}+1$ vertices contains an ordered k-vertex hypergraph consisting of a red and a blue monotone path that are vertex disjoint except for the common end-vertices.

Applying the state of art SAT solver, we refute the conjecture of Peters and Szekeres. We also apply techniques of Erdős, Tuza, and Valtr to refine the Erdős–Szekeres conjecture in order to tackle it with SAT solvers.

Keywords: Erdős–Szekeres conjecture, SAT solver, convex position, Ramsey number

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1 Introduction

The *Erdős–Szekeres theorem* [5] is, without an exaggeration, one of the most important results in Ramsey theory. It says that for every integer $k \geq 2$ there is a least number $\mathrm{ES}(k)$ such that every set of $\mathrm{ES}(k)+1$ points in the plane in *general position* (no three points lie on a common line and all x-coordinates are distinct) contains k points in convex position. Erdős and Szekeres [5] proved an upper bound $\mathrm{ES}(k) \leq \binom{2k-4}{k-2}$ and posed the *Erdős–Szekeres conjecture*, stating that $\mathrm{ES}(k) = 2^{k-2}$ for every $k \geq 2$.

In the 1960s, Erdős and Szekeres [6] supported their conjecture with the lower bound $\mathrm{ES}(k) \geq 2^{k-2}$. Despite many efforts, the Erdős–Szekeres conjecture is known to hold only for $k \leq 6$ and is open for $k > 6$. The case $k = 6$ was proven by Peters and Szekeres [10] who carried out an exhaustive computer search.

Applying the state of art SAT solver, we refute a natural strengthening of the Erdős–Szekeres conjecture introduced by Peters and Szekeres [10]. We also apply techniques of Erdős et al. [7] to refine the Erdős–Szekeres conjecture in order to tackle it with SAT solvers. Despite this attempt, the Erdős–Szekeres conjecture remains open.

2 The Peters–Szekeres conjecture

Points $p_1, \ldots, p_k \in \mathbb{R}^2$ with increasing x-coordinates form a *k-cap* if the slopes of the lines $\overline{p_1p_2}, \ldots, \overline{p_{k-1}p_k}$ are decreasing. The points $p_1, \ldots, p_k$ form a *k-cup* if the slopes are increasing. For integers $a, u \geq 2$, let $\mathrm{N}(a,u)$ be the maximum size of a set of points in the plane in general position with no a-cap and no u-cup. Note that vertices of every k-cap and k-cup are in convex position and thus we have $\mathrm{ES}(k) \leq \mathrm{N}(k,k)$.

Erdős and Szekeres [5] proved the bound $\mathrm{ES}(k) \leq \binom{2k-4}{k-2}$ by showing

$$\mathrm{N}(a,u) = \binom{a+u-4}{a-2} = \binom{a+u-4}{u-2} \tag{1}$$

for all integers $a, u \geq 2$. Recently, Fox et al. [8] suggested the following framework for studying the Erdős–Szekeres theorem in terms of ordered hypergraphs. Let K^3_N be the complete 3-uniform hypergraph with the vertex set $\{1, \ldots, N\}$ ordered by a linear ordering $<$. For vertices $v_1 < \cdots < v_k$ of K^3_N, the edges $\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}, \ldots, \{v_{k-2}, v_{k-1}, v_k\}$ form a *monotone path of length k*, or *k-path* for short. A *coloring of K^3_N* is a mapping that assigns either a red or a blue color to every edge of K_N^3. Let $\widehat{\mathrm{N}}(a,u)$ be the maximum number N such that there is a coloring of K_N^3 with no red a-path and no blue u-path.

We now observe that $\mathrm{N}(a,u) \leq \widehat{\mathrm{N}}(a,u)$ for all integers $a,u \geq 2$. Let P be a point set in the plane in general position. We color every triple T of points from P, ordered by x-coordinates, red if T is oriented clockwise and blue if T is oriented counterclokwise. Every coloring of K_N^3 obtained in this way from some point set of size N is called *realizable*. The inequality follows, as, for every $k \geq 3$, k-caps and k-cups in P are in one-to-one correspondence with red and blue, respectively, k-paths in the realizable coloring obtained from P.

A straightforward generalization of the proof of (1) gives $\widehat{\mathrm{N}}(a,u) = \mathrm{N}(a,u)$ for all $a,u \geq 2$. Peters and Szekeres [10] conjectured that a similar phenomenon occurs for the Erdős–Szekeres conjecture. We state this formally.

If P is a point set in the plane in general position, then every k-tuple of points from P in convex position is a union of an a-cap and a u-cup with common endpoints where a and u are some integers satisfying $a+u-2=k$. Using this fact, Peters and Szekeres [10] generalized the notion of a convex position to the hypergraph setting as follows. For $k \geq 2$, an ordered 3-uniform hypergraph H on k vertices is called a *(convex) k-gon* if H consists of a red and a blue monotone path that are vertex disjoint except for the common end-vertices. We allow paths in H with two vertices and no edges. Note that there are 2^{k-2} pairwise non-isomorphic k-gons for every $k \geq 2$. Let $\widehat{\mathrm{ES}}(k)$ be the maximum number N such that there is a coloring of K_N^3 with no k-gon.

If P is a set of points in the plane in general position, then k-tuples of points from P in convex position are in one-to-one correspondence with k-gons in the realizable coloring of $K_{|P|}^3$ obtained from P. Thus we have $2^{k-2} \leq \mathrm{ES}(k) \leq \widehat{\mathrm{ES}}(k)$ for every $k \geq 2$. On the other hand, every monochromatic k-path is a k-gon, thus from $\widehat{\mathrm{N}}(a,u) = \binom{a+u-4}{a-2}$ we obtain $\widehat{\mathrm{ES}}(k) \leq \binom{2k-4}{k-2}$.

For $2 \leq k \leq 5$, Peters and Szekeres [10] showed $\widehat{\mathrm{ES}}(k) = 2^{k-2}$. For $k=5$ this was shown by an exhaustive computer search. Peters and Szekeres conjectured that this equality is satisfied for every $k \geq 2$. We call this conjecture the *Peters–Szekeres conjecture*. As our main result we refute this conjecture.

Theorem 2.1 *We have* $\widehat{\mathrm{ES}}(7) > 32$ *and* $\widehat{\mathrm{ES}}(8) > 64$.

3 The Erdős–Szekeres conjecture revisited

In this section we introduce an equivalent version of the Peters–Szekeres conjecture that we use later in a search for a counterexample. Our approach is based on the following equivalent version of the Erdős–Szekeres conjecture introduced by Erdős et al. [7].

For integers a, u, k that satisfy $2 \leq a, u \leq k \leq a + u - 2$, let $\mathrm{N}(a, u, k)$ be the maximum number N such that there is a set of N points in the plane in general position with no a-cap, no u-cup, and no k points in convex position.

Conjecture 3.1 (Erdős et al.[7]) *For all integers a, u, k with $2 \leq a, u \leq k \leq a + u - 2$, we have $\mathrm{N}(a, u, k) = \sum_{i=k-a+2}^{u} \mathrm{N}(i, k + 2 - i)$. In particular, $\mathrm{N}(a, u, k) = \sum_{i=k-a+2}^{u} \binom{k-2}{i-2}$.*

Erdős et al. [7] showed that Conjecture 3.1 is equivalent with the Erdős-Szekeres conjecture and proved the inequality $\mathrm{N}(a, u, k) \geq \sum_{i=k-a+2}^{u} \binom{k-2}{i-2}$ for all a, u, k with $2 \leq a, u \leq k \leq a + u - 2$.

The best known upper bound for $\mathrm{N}(a, u, k)$ is $\mathrm{N}(a, u, k) \leq \binom{a+u-4}{a-2}$, which is obtained from the trivial estimate $\mathrm{N}(a, u, k) \leq \mathrm{N}(a, u)$. For $k = a + u - 2$, Conjecture 3.1 is true by (1), as $\mathrm{N}(a, u, k) = \mathrm{N}(a, u)$ in this case. For $k = a + u - 3$, Conjecture 3.1 says $\mathrm{N}(a, u, k) = \binom{a+u-5}{u-3} + \binom{a+u-5}{u-2} = \binom{a+u-4}{u-2}$, which is again true.

The gap between known bounds for $\mathrm{N}(a, u, k)$ appears first for $k = a+u-4$. By a more careful analysis of this first nontrivial case, we improve the best known upper bound by one for the case $a = 4$. The proof is omitted.

Proposition 3.2 *For every integer $k \geq 3$, we have $\mathrm{N}(4, k, k) \leq \binom{k}{2} - 1$.*

Now, we introduce a version of Conjecture 3.1 for the hypergraph setting. For integers a, u, k that satisfy $2 \leq a, u \leq k \leq a + u - 2$, let $\widehat{\mathrm{N}}(a, u, k)$ be the maximum number N such that there is a coloring of K_N^3 with no red a-path, no blue u-path, and no k-gon.

Conjecture 3.3 *For all integers a, u, k with $2 \leq a, u \leq k \leq a + u - 2$, we have $\widehat{\mathrm{N}}(a, u, k) = \sum_{i=k-a+2}^{u} \widehat{\mathrm{N}}(i, k + 2 - i) = \sum_{i=k-a+2}^{u} \binom{k-2}{i-2}$.*

A straightforward generalization of the approach of Erdős et al. [7] gives the following statement whose proof is again omitted.

Proposition 3.4 *Conjecture 3.3 is equivalent with the Peters–Szekeres conjecture.*

The main profit gained by considering $\widehat{\mathrm{N}}(a, u, k)$ is that Conjecture 3.3 is, in a certain sense, finer than the Peters–Szekeres conjecture. This allows us to employ an exhaustive computer search for larger values of k in order to find a coloring of K_N^3 with no red a-path, no blue u-path, and no k-gon for some suitable integers a, u, and $N > \sum_{i=k-a+2}^{u} \binom{k-2}{i-2}$. This will disprove Conjecture 3.3 and, by Proposition 3.4, the Peters–Szekeres conjecture.

The exhaustive search for extremal colorings is performed by SAT solvers. We use a SAT encoding of the following decision problem: for given integers $a, u, k, N \geq 3$, is there a coloring of K_N^3 with no red a-path, no blue u-path, and no k-gon?

4 Results

In our experiments we use the *Glucose* SAT solver [1], the winner of the *certified UNSAT* category of the SAT 2013 competition [2]. All experiments were conducted on a computer equipped with Intel Xeon E5-1620 CPU running at 3.60GHz and 63GB of RAM.

We found a coloring c of K_{17}^3 with no red 4-path and no 7-gon. This refutes Conjecture 3.3 for $a = 4$ and $u = k = 7$. It follows from the proof of Proposition 3.4 that the coloring c can be extended to a coloring of K_{33}^3 with no 7-gon, therefore we refute the Peters–Szekeres conjecture as well. Our experiments showed that every coloring of K_{18}^3 contains either a red 4-path or a 7-gon, i. e., $\widehat{\mathrm{N}}(4, 7, 7) = 17$.

By running additional tests, we obtained further counterexamples to Conjecture 3.3. We found colorings that give $\widehat{\mathrm{N}}(5, 6, 7) \geq 26$, $\widehat{\mathrm{N}}(5, 7, 7) \geq 27$, $\widehat{\mathrm{N}}(6, 6, 7) \geq 31$, $\widehat{\mathrm{N}}(6, 7, 7) \geq 32$, and even $\widehat{\mathrm{N}}(7, 7, 7) \geq 33$. We also obtained the bound $\widehat{\mathrm{N}}(4, 8, 8) \geq 23$ that provides a counterexample to Conjecture 3.3 and to the Peters–Szekeres conjecture for $k = 8$. For $k > 8$, the input formulas become too large for the SAT solver, even in the case $a = 4$ and $u = k = 9$.

Our experiments verify Conjecture 3.3 for $k = 6$ and for all possible values of a and u, except for the case $a = u = k$. In this case the solver did not terminate on 17 vertices even after 266 hours of computation.

We also run tests to explore the validity of Conjecture 3.1. Our approach is based on a restriction of the setting of Conjecture 3.3 to *pseudolinear* colorings. A coloring c' of K_N^3 is pseudolinear if every 4-tuple of vertices of K_N^3 induces a realizable coloring of K_4^3 in c'. Clearly, every realizable coloring is pseudolinear.

Peters and Szekeres [10] call pseudolinear colorings 'signatures that satisfy geometric constraints'. However, we feel that the term 'pseudolinear coloring' is more accurate, as such colorings of K_N^3 are in one-to-one correspondence with *pseudolinear x-monotone drawings* of K_N, see [3, Theorem 3.2].

Considering only pseudolinear colorings in our experiments, we verified

Conjecture 3.1 in the cases $a = 4$, $u = k = 7$ and $a = 4$, $u = k = 8$. That is, we have $\mathrm{N}(4, 7, 7) = 16$ and $\mathrm{N}(4, 8, 8) = 22$. For pseudolinear colorings, all our results matched the values from Conjecture 3.1. All colorings obtained by our experiments can be found in [4].

5 Final remarks

The following strengthening of the Erdős–Szekeres conjecture, introduced by Peters and Szekeres [10], remains open: for every $k \geq 2$, is it true that every pseudolinear coloring of K_N^3 with $N = 2^{k-2} + 1$ contains a k-gon? Similarly, Goodman and Pollack [9] conjectured that for every $k \geq 2$ the number $\mathrm{ES}(k)$ equals the maximum N for which there is a pseudolinear coloring of K_N^3 with no k-gon. Note that the Goodman-Pollack conjecture might be true even if the previous strengthening is not.

None of the counterxamples for Conjecture 3.3 is pseudolinear. If there was a pseudolinear coloring c that refutes Conjecture 3.3, then we could use the proof of Proposition 3.4 and extend c to a counterexample to the strengthening above. If c was realizable, then it would give a counterexample even to the Erdős–Szekeres conjecture.

Another possible direction for further research is to improve the bounds for $\widehat{\mathrm{ES}}(k)$ and, possibly, to recognize some structure behind the colorings that we found. For sufficiently large k, this could lead to a general construction of colorings of K_N^3 with no k-gon for $N > 2^{k-2} + 1$.

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