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Strong Turán stability

Mykhaylo Tyomkyn [1,2]

School of Mathematics
University of Birmingham
Birmingham B15 2TT, United Kingdom

Andrew J. Uzzell [3]

Department of Mathematics
University of Nebraska-Lincoln
Lincoln NE 68588-0130, USA

Abstract

We study the behaviour of K_{r+1}-free graphs G of almost extremal size, that is, typically, $e(G) = ex(n, K_{r+1}) - O(n)$. We show that such graphs must have a large amount of symmetry. In particular, if G is saturated, then all but very few of its vertices must have twins. As a corollary, we obtain a new proof of a theorem of Simonovits on the structure of extremal graphs with $\omega(G) \leq r$ and $\chi(G) \geq k$ for fixed $k \geq r \geq 2$.

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[2] Email: m.tyomkyn@bham.ac.uk

[3] Email: andrew.uzzell@math.unl.edu

1 Introduction

Let $T_{n,r}$ denote the Turán graph on n vertices with r partition classes of size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$ each, and put $t_{n,r} := e(T_{n,r})$. From Turán's theorem we know that $t_{n,r}$ maximises the size of a K_{r+1}-free graph of order n. One of the best known extensions of Turán's theorem is the Erdős–Simonovits stability theorem, which says, in particular, that a K_{r+1}-free graph on n vertices and $t_{n,r} - o(n^2)$ edges can be turned into $T_{n,r}$ by adding or removing $o(n^2)$ edges. To phrase it qualitatively, a K_{r+1}-free graph whose size is close to being extremal looks essentially like the extremal graph. This behaviour became known as *stability*, and has been extensively studied in various structures.

In this paper we are concerned with different aspects of Turán stability. More concretely, we will study K_{r+1}-free graphs G with $e(G) = t_{n,r} - O(n)$ or $e(G) = t_{n,r} - O(n \log n)$. This is much closer to the Turán threshold than the range of the Erdős–Simonovits stability theorem and allows to observe different aspects of stability. Our results can therefore be viewed as a part of a larger programme of studying the 'phase transition' of K_{r+1}-free graphs near the Turán threshold that has been emphasized by Simonovits.

First we give a new proof of a theorem that was first explicitly proved by Brouwer [3] (although implicitly it follows from earlier work of Simonovits [7], and in the case $r = 2$ it was proved by Andrásfai, Erdős, and Gallai [4]), and that has been re-discovered several times [1,5,6].

Let

$$h(n,r) = \begin{cases} t_{n,r} - \left\lfloor \frac{n}{r} \right\rfloor + 1, & n \geq 2r+1, \\ t_{n,r} - 2, & r+3 \leq n \leq 2r, \end{cases} \tag{1}$$

and note that the second case is vacuous if $r = 2$.

Theorem 1.1 *If $n \geq r+3$, then every K_{r+1}-free graph of order n and size at least $h(n,r)+1$ is r-colourable.*

Unlike the Erdős–Simonovits theorem, which says that a K_{r+1}-free graph on sufficiently many edges is *approximately* r-partite, this theorem gives a condition for a K_{r+1}-free graph to *actually* be r-partite.

A natural generalisation of Theorem 1.1 would be to find the maximal number of edges in a graph G with $|V(G)| = n$, $\omega(G) = r$ and $\chi(G) \geq k$. It is easy to see that the extremal number is of order $t_{n,r} - O(n)$: for instance, take the disjoint union of a Turán graph $T_{n',r}$ and a finite size graph G' with $\omega(G') = r$ and $\chi(G') \geq k$. Determining the constant in the linear term asymptotically as $k \to \infty$ is less interesting in its own right, since it is closely related to the asymptotic behaviour of the Ramsey numbers $R(r+1,k)$.

A much more interesting problem is the structure of the extremal graphs. One simple way to construct such graphs (more efficiently than the trivial construction given above) is the following: take a finite size graph G' with $\omega(G') = r$ and $\chi(G') = k$, and blow up an r-clique of G' in a way that would maximise the number of edges. Let us call a graph (or, more precisely, a graph sequence) *simple* if it is a blow-up of a bounded order graph. It is natural to ask whether the extremal graph must be simple. This was answered in the affirmative by Simonovits for $r = 2$ in [8] and (as a part of a more general result) for arbitrary r in [9].

We suggest a new generalisation of Theorem 1.1, namely the study of *maximal* (or *saturated*) K_{r+1}-free graphs on many edges; note that the extremal graph for a given chromatic number is a special case. In the spirit of Simonovits' theorem we prove sharp bounds on how large $e(G)$ should be in order for G to be simple. Perhaps surprisingly, the thresholds for $r = 2$ and for $r \geq 3$ turn out to be substantially different, with the proof being very short in the former case and more involved in the latter.

Theorem 1.2 *For every $c > 0$ every 3-saturated graph G on n vertices with $e(G) > t_{n,2} - cn$ is simple.*

Let $r \geq 3$. For every $\varepsilon > 0$ every $(r+1)$-saturated graph G on n vertices with $e(G) > t_{n,r} - (2-\varepsilon)n/r$ is simple.

Taking this study further, we obtain a sharp threshold for a maximal K_{r+1}-free graph to have a single pair of twin vertices (vertices with identical neighbourhoods). Clearly, this threshold has to be lower than the bound in Theorem 1.2. We consider the following theorem to be the main result of this paper.

Theorem 1.3 *For every $r \geq 2$ there exists a constant $c > 0$ such that every sufficiently large $(r+1)$-saturated graph G with $e(G) \geq t_{n,r} - cn \log n$ has a pair of twin vertices.*

Note that unlike the previous theorem, in this case the bounds are similar for all values of r, though the proof is still much shorter in the case $r = 2$. As a corollary of Theorem 1.3, we obtain a new simple proof of the aforementioned theorem of Simonovits, formally stated as follows.

Theorem 1.4 *For each $r \geq 2$ and each $k \geq r$, there exists $m(k,r)$ such that if G is an extremal K_{r+1}-free graph with chromatic number at least k, then G is a blow-up of a graph G' with $|G'| \leq m(k,r)$.*

In other words, for every r and k, the sequence of extremal graphs G for

$\omega(G) \leq r$ and $\chi(G) \geq k$ is simple.

2 Clique-saturated graphs

We will frequently apply the following simple corollary of the Andrásfai-Erdős-Sós Theorem [2].

Lemma 2.1 *There exists a function $g(r,c)$ such that the vertex set of every K_{r+1}-free graph G with $e(G) \geq t_{n,r} - cn$ can be split into a set F with $|F| \leq g(r,c)$ and an r-partite graph $V \setminus F$.*

2.1 Finite-size reductions

The unique largest $(r+1)$-saturated graph, the Turán graph $T_{n,r}$, is a balanced blow-up of K_r. Moreover, by Theorem 1.1 a K_{r+1}-free graph G that has more than $t_{n,r} - n/r + 1$ edges is r-chromatic. Hence, if G is $(r+1)$-saturated, then all edges between different partition classes must be present, so G is complete r-partite (possibly with unbalanced colour classes), i.e. it is another blow-up of K_r. It is natural to ask: if we continue to decrease $e(G)$, how long will G remain a blow-up of a finite order graph? In other words, what is the largest function $f_r(n)$ such that every $(r+1)$-saturated graph with at least $t_{n,r} - f_r(n)$ edges is a blow-up of a graph whose order does not depend on n?

We begin by proving Theorem 1.2 in the case $r = 2$.

Theorem 2.2 *For every $c \geq 0$ there exists $m_2(c)$ such that every 3-saturated graph G on n vertices with $e(G) > t_{n,2} - cn$ is a blow-up of some (triangle-free) graph H with $|H| \leq m_2$.*

Proof. If G is bipartite, then it must be complete bipartite, and we are done. If G is not bipartite, then by Lemma 2.1 it is composed of a large bipartite graph $G_b = (U, W, E_b)$ and an exceptional vertex set V_e with $|V_e| \leq g(r,c)$. Now, partition the vertices of U and W according to their V_e-neighbourhood: for every $X \subset V_e$, define

$$U_X := \{u \in U : N_{V_e}(u) = X\},$$

and W_X analogously. Take any $u \in U$ and $w \in W$. Let $X = N_{V_e}(u)$ and $Y = N_{V_e}(w)$, so that $u \in U_X$ and $w \in W_Y$. If $X \cap Y = \emptyset$, then u and w must be adjacent, since G is 3-saturated. On the other hand, if $X \cap Y \neq \emptyset$, there can be no edge between u and w, as it would create a triangle. Hence, the neighbourhoods of u and w are completely determined by their

V_e-neighbourhoods, meaning that two vertices u_1, $u_2 \in U_X$ for any given X are twins (the same holds in W). Since there are at most $2^{|V_e(G)|}$ possible V_e-neighbourhoods, we conclude that G has at most

$$|V_e(G)| + 2 \cdot 2^{|V_e(G)|}$$

twin classes. Thus, the statement of the theorem holds with $m_2(c) = g(r, c) + 2 \cdot 2^{g(r,c)}$.

□

Note that extremal triangle-free ($\geq k$)-chromatic graphs are in particular 3-saturated. As was mentioned in the Introduction, it is easy to construct a triangle-free, ($\geq k$)-chromatic graph with $t_{n,2} - c_k n$ edges. Thus, as an immediate Corollary of Theorem 2.2 we obtain Theorem 1.4 (Simonovits' Theorem) for $r = 2$.

Corollary 2.3 *For each $k \geq 2$ there exists a constant $m(k, 2)$ such that if G is an extremal triangle-free ($\geq k$)-chromatic graph on n vertices, then G is a blow-up of a graph G' with $|G'| \leq m(k, 2)$.*

The following construction will demonstrate that the bound of Theorem 2.2 is sharp in the following sense: given a function $f(n)$ that tends to infinity (no matter how slowly), there exist 3-saturated graphs G with $e(G) = t_{n,2} - nf(n)$ yet with an unbounded number of twin classes.

Example 2.4 We may assume that $f(n) < \frac{\log_2 n}{2}$. Let S be a set of $f(n)$ vertices, let U and W be disjoint sets of $2^{f(n)}$ vertices each, and divide the rest of the vertices equally into two sets U' and W'. Give different vertices of U distinct neighbourhoods in S, and similarly for vertices in W: for each $I \subset S$, let u_I be the vertex in U with $N_S(u_I) = I$, and define w_I similarly. Join u_I and w_J if and only if I and J are disjoint. Finally, add all edges between U' and W', between U' and W, and between U and W'. It is not hard to see that the resulting graph G is 3-saturated. Also, G has at least $2^{f(n)+1} + f(n)$ distinct neighbourhoods.

Since $f(n) < \frac{\log_2 n}{2}$, we obtain

$$\begin{aligned} e(G) &> |U'||W'| + |U'||W| + |U||W'| > t_{n-f(n),2} - 2^{2f(n)} \\ &> t_{n,2} - \frac{nf(n)}{2} - 2^{2f(n)} > t_{n,2} - nf(n), \end{aligned}$$

as claimed.

Now let us consider the case $r \geq 3$. Perhaps surprisingly, the analogue of Theorem 2.2 does not hold here, as the following construction shows.

Example 2.5 Let $n \in \mathbb{N}$, let $m = (1/2)\log_2 n$, and let $M = \binom{m}{m/2}$; note that $M < \sqrt{n}$. Take the Turán graph $T_{n-1,r}$ where $V_1, \ldots, V_r$ denote its partition classes. Let $W_1 \subset V_1$, $W_2 \subset V_2$ and $W_3 \subset V_3$ with $|W_1| = M$ and $|W_2| = |W_3| = m$. Introduce a new vertex v to G and join it to all of the vertices of the W_i and to all of the vertices of V_j for $j \neq \{1, 2, 3\}$. Remove all edges between different W_i. The resulting graph G' satisfies

$$e(G') \geq t_{n-1,r} - 2mM - m^2 + \left\lfloor \frac{r-3}{r}(n-1) \right\rfloor + M + 2m = t_{n,r} - \frac{2n}{r} + o(n).$$

Now add a matching between W_2 and W_3 and for each $w \in W_1$ we select a subset $U_w \subset W_2$ of size $m/2$ such that different vertices of W_1 receive distinct subsets. Connect w to U_w in W_2 and to $W_3 \setminus N_{W_3}(U_w)$ in W_3.

It is easy to check that obtained graph G is $r+1$ saturated. Moreover, no vertices in W_1 are twins, so G has an unbounded number of twin classes.

Given $r \geq 3$, let c_r be the supremum of the numbers c such that every $(r+1)$-saturated graph G with $e(G) > t_{n,r} - cn$ has a bounded number of twin classes.

Observe that Theorem 1.1 and Example 2.5 imply that $1/r \leq c_r \leq 2/r$. Our next result shows that $c_r = 2/r$ holds for all $r \geq 3$, thereby completing the proof of Theorem 1.2.

Theorem 2.6 *For every $r \geq 3$ and every $\varepsilon > 0$ there exists $m_r(\varepsilon)$ such that every $(r+1)$-saturated graph G with $e(G) > t_{n,r} - (2-\varepsilon)n/r$ is a blow-up of some (K_{r+1}-free) graph H with $|H| \leq m_r$.*

2.2 Twin-free saturated graphs

What is the largest number of edges that an $(r+1)$-saturated graph G can have if *no two* vertices of G are twins?

In the case $r = 2$ we have the following result, which is proved along the same lines as Theorem 2.2.

Proposition 2.7 *For each $\varepsilon > 0$ every sufficiently large 3-saturated graph with $e(G) > n^2/4 - (1/10 - \varepsilon)n\log_2 n$ contains a pair of twins.*

On the other hand, it can be shown that Proposition 2.7 is best possible up to a constant factor in the $n\log_2 n$-term.

We omit the proof of Theorem 1.3 for every $r \geq 3$ here for space reasons, but let us give a construction illustrating that its bound is is best possible up to the value of the constant c.

Example 2.8 For n sufficiently large, we construct a twin-free, $(r+1)$-saturated graph on n vertices as follows. Let H be the disjoint union of $T_{n-r,r}$ and r isolated vertices $u_1, \ldots, u_r$. Let $V_1, \ldots, V_r$ denote the colour classes of the copy of $T_{n-r,r}$.

Let m be a quantity to be defined later and let $M = \binom{m}{m/2}$. We partition $V_1 \cup \cdots \cup V_r$ into three families of sets $\{W_1^{(i)}\}_{i=1}^r$, $\{W_2^{(i)}\}_{i=1}^r$ and $\{W_3^{(i)}\}_{i=1}^r$ such that for each i, we have $W_1^{(i)} \subset V_i$, $W_2^{(i)} \subset V_{i+1}$ and $W_3^{(i)} \subset V_{i+2}$ (where the addition is modulo r), as well as that $|W_1^{(i)}| = M$ and $|W_2^{(i)}| = |W_3^{(i)}| = m$. It follows that

$$n = r(M + 2m + 1). \tag{2}$$

Because $m = o(M)$, (2) implies that

$$M \sim n/r, \tag{3}$$

which in turn implies that

$$m \sim \log_2 n. \tag{4}$$

Now we modify H in order to make it twin-free and maximal K_{r+1}-free. For each i, $i = 1, \ldots, r$, we modify $H[W_1^{(i)} \cup W_2^{(i)} \cup W_3^{(i)}]$ as in Example 2.5. Then we connect u_i to all vertices of $W_1^{(i)} \cup W_2^{(i)} \cup W_3^{(i)}$ and to all vertices of each V_k, $k \notin \{i, i+1, i+2\}$ (mod r). Finally, we greedily add edges among the u_i.

Let G denote the resulting graph. It is easy to check that G is $(r+1)$-saturated, twin-free, and that $e(G) = t_{n,r} - n \log_2 n + O(n)$.

2.3 Large complete subgraphs

Here we consider another way in which an $(r+1)$-saturated graph may be 'close' to $T_{n,r}$, namely, having a large complete r-partite subgraph. We have shown that if $r \geq 3$ and if c is large enough, then there exist $(r+1)$-saturated graphs with $t_{n,r} - cn$ edges that are not simple. However, every 4-saturated graph with at least this many edges must contain a large complete tripartite subgraph.

Theorem 2.9 *For every $c > 0$ every 4-saturated graph G with $e(G) > t_{n,3} - cn$ contains a complete tripartite graph on $(1 - o(1))n$ vertices.*

The proof does not seem to extend straightforwardly to the general case, which we leave as an open problem.

Problem 2.10 *Given $r \geq 4$ and $c > 0$, what is the order of the largest complete r-partite subgraph in a given $(r+1)$-saturated graph with $t_{n,r} - cn$?*

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