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Extremal problems for colorings of simple hypergraphs and applications

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Abstract

The paper deals with extremal problems concerning colorings of hypergraphs. By using a random recoloring algorithm we show that any n-uniform simple hypergraph H with maximum edge degree at most $\Delta(H) \leq c \cdot nr^{n-1}$, is r-colorable, where $c > 0$ is an absolute constant. As an application of our proof technique we establish a new lower bound for the Van der Waerden number $W(n, r)$.

Keywords: Colorings of hypergraphs, simple hypergraphs, Van der Waerden numbers.

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1 Introduction

A hypergraph is a pair (V, E) where V is a set, called the *vertex set* of the hypergraph and E is a family of subsets of V, whose elements are called the *edges* of the hypergraph. A hypergraph is *n-uniform* if every of its edges contains exactly n vertices. In a fixed hypergraph, the *degree of a vertex v* is the number of edges containing v, the *degree of an edge e* is the number of other edges of the hypergraph which have nonempty intersection with e. The maximum edge degree of hypergraph H is denoted by $\Delta(H)$.

An r-coloring of hypergraph $H = (V, E)$ is a mapping from the vertex set V to the set of r colors, $\{0, \ldots, r-1\}$. A coloring of H is called *proper* if it does not create monochromatic edges (i.e. every edge contains at least two vertices which receives different colors). A hypergraph is said to be *r-colorable* if there exists a proper r-coloring of that hypergraph. Finally, the *chromatic number* of hypergraph H is the minimum r such that H is r-colorable.

The first quantitative relation between the chromatic number and the maximum edge degree in an uniform hypergraph was obtained by Erdős and Lovász in their classical paper [1]. They proved that if H is an n-uniform hypergraph and

$$\Delta(H) \leq \frac{1}{4} r^{n-1}, \tag{1}$$

then H is r-colorable. However the bound (1) is not tight. The restriction on the maximum edge degree was successively improved in a series of papers. In connection with the classical problem related to Property B, Radhakrishnan and Srinivasan [2] proved that any n-uniform hypergraph H with

$$\Delta(H) \leq 0,17\sqrt{\frac{n}{\ln n}}2^{n-1} \tag{2}$$

is 2-colorable. The generalization of the above result was found by Cherkashin and Kozik [3]. They showed that, for a fixed $r \geq 2$ there exists a positive constant $c(r)$ such that for all large enough $n > n_0(r)$, if H is an n-uniform hypergraph and

$$\Delta(H) \leq c(r)\left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}} r^{n-1}, \tag{3}$$

then H is r-colorable.

Extremal problems concerning colorings of hypergraphs are closely connected to the classical questions of Ramsey theory. The hypergraphs appearing in these challenging problems are very close to be simple. Recall that hypergraph (V, E) is called *simple* if every two of its distinct edges share at most one vertex, i.e. for any $e, f \in E$, $e \neq f$, $|e \cap f| \leq 1$. It is natural to expect that it is easier to color simple hypergraphs and that the bounds (1)–(3) can be improved. In [4] Kostochka and Kumbhat proved that for arbitrary $\varepsilon > 0$ and $r \geq 2$, there exists $n_0 = n_0(\varepsilon, r)$ such that if $n > n_0$ and an n-uniform simple hypergraph H satisfies

$$\Delta(H) \leq n^{1-\varepsilon} r^{n-1}, \tag{4}$$

then H is r-colorable. Since $\varepsilon > 0$ is arbitrary in (4) then, of course, it can be replaced by some infinitesimal function $\varepsilon = \varepsilon(n) > 0$, for which $\varepsilon(n) \to 0$ as $n \to \infty$. Few papers were devoted to the problem of estimating the order of its growth. Recently the progress was made independently by Kozik [5] and by Kupavskii and Shabanov [6], who proved respectively that the bounds

$$\Delta(H) \leq c\,\frac{n}{\ln n} r^{n-1} \quad \textit{and} \quad \Delta(H) \leq c\,\frac{n(\ln\ln n)^2}{\ln n} r^{n-1}$$

guarantee r-colorability of a simple n-uniform hypergraph H.

The main result of the current paper completely removes the factor $n^{-\varepsilon}$ from the bound (4).

Theorem 1.1 *There exists a positive constant α such that for every $r \geq 2$, and every $n \geq 3$, any simple n-uniform hypergraph with maximum edge degree at most $\alpha \cdot n\, r^{n-1}$ is r-colorable.*

Note that in comparison with (4), Theorem 1.1 holds for any $r \geq 2$, not only for fixed values of r. Methods used in the proof of Theorem 1.1 can be used to address analogous problems in other classes of hypergraphs. We present such an extension concerning hypergraphs of arithmetic progressions over integers. That allows us to derive a new lower bound for the Van der Waerden number. Recall that the Van der Waerden number $W(n, r)$ is the minimum N such that in any r-coloring of integers $\{1, \dots, N\}$ there exists a monochromatic arithmetic progression of length n.

Theorem 1.2 *There exists positive β such that for every $r \geq 2$ and $n \geq 3$, we have*

$$W(n, r) \geq \beta r^{n-1}.$$

That improves over the bound of Szabó [7] of the order $n^{-|o(1)|} r^{n-1}$ and over recent bounds by Kozik [5] and by Kupavskii and Shabanov [6].

2 Ideas of the proof

In this section we describe a coloring algorithm which underlies the proof of our main result. Let $H = (V, E)$ be an n-uniform hyper graph and r be a number of colors. The algorithm follows the general principle of the *random recoloring method*: for given non-proper coloring it tries to recolor a small number of vertices from the monochromatic edges to make the coloring proper.

The algorithm is parameterized by $p \in (0, 1/2)$ and gets two inputs: first is an *initial coloring* $c : V \to \{0, \ldots, r-1\}$ of the hypergraph, second is an injective function $\sigma : V \to [0, 1]$, called *weight* assignment. For every vertex v, color $c(v)$ assigned to it by the initial coloring is called *the initial color* of v. The value $\sigma(v)$ is called the *weight* of v. A vertex v is called *free* if $\sigma(v) \leq p$. Recall that an edge is monochromatic w.r.t. some coloring if all its vertices get the same color. In any set of vertices the *first vertex* is the vertex v with the minimum weight, i.e. the minimum value of $\sigma(v)$. We use a succinct notation $(n)_r$ to denote the value of $n \pmod r$.

Algorithm 1. Input: $c : V \to \{0, \ldots, r-1\}$, $\sigma : V \to (0, 1]$ injective
While *there exists a monochromatic edge whose first non–recolored vertex v is free*
Do $c(v) \to (c(v) + 1)_r$ (i.e. v is recolored with $(c(v) + 1)_r$)
Return c

Note that during the evaluation of the algorithm every vertex changes its color at most once, therefore the procedure always stops. The main result of the paper concerning colorings of simple hypergraphs can be reformulated by using Algorithm 1 as follows.

Theorem 2.1 *Let $H = (V, E)$ be an n-uniform simple hypergraph with $\Delta(H) \leq \alpha \cdot n\, r^{n-1}$ for some constant $\alpha > 0$. If the input (c, σ) is chosen uniformly at random then Algorithm 1 produces a proper r-coloring for H with positive probability.*

In order to analyze the situations in which the coloring returned by the algorithm is not proper for a fixed hypergraph $H = (V, E)$, we introduce the notion of h-tree. An *h-tree* is a rooted tree labelled according to the following rules:

(i) each tree node x is labelled by an edge $e(x)$ of the hypergraph H,

(ii) each tree edge f is labelled by a vertex $v(f)$ of the hypergraph H,

(iii) for a tree edge $f = (x_1, x_2)$ we have $e(x_1) \cap e(x_2) \ni v(f)$.

So, the nodes of an h-tree correspond to the edges of H and two edges that are neighbors in the h-tree should intersect in the hypergraph.

Suppose now that the algorithm produces a non-proper coloring and edge e is monochromatic in the final coloring. We very briefly describe how to construct an h-tree that *witnesses* the failure of the algorithm. If during the evaluation of the algorithm some vertex v is recolored, then it should be the first non-recolored free vertex of some edge f that at that moment of the procedure was monochromatic. In this case vertex v is said to *blame* the edge f and for every free vertex, which has been recolored during the evaluation of the algorithm, we choose one edge to be blamed. We say that an edge f_1 blames an edge f_2 if f_1 contains a vertex that blames f_2. Starting with the monochromatic edge e, we use the blaming relationship to find an h-tree with special properties. The exact estimates for the probability that some fixed h-tree witnesses a failure of the algorithm depend on the real relations between the edges of H that label nodes of the h-tree. In every case we estimate the probability of the corresponding event and then the number of considered configurations containing a fixed vertex of the hypergraph.

Finally, we show that for appropriately chosen constant α and parameter p, we can apply a special variant of the Lovász Local Lemma (see [5]) to prove that all the possible bad events can be avoided with positive probability.

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