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The spectral excess theorem for distance-regular graphs having distance-d graph with fewer distinct eigenvalues

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Abstract

Let Γ be a distance-regular graph with diameter d and Kneser graph $K = \Gamma_d$, the distance-d graph of Γ. We say that Γ is partially antipodal when K has fewer distinct eigenvalues than Γ. In particular, this is the case of antipodal distance-regular graphs (K with only two distinct eigenvalues), and the so-called half-antipodal distance-regular graphs (K with only one negative eigenvalue). We provide a characterization of partially antipodal distance-regular graphs (among regular graphs with d distinct eigenvalues) in terms of the spectrum and the mean number of vertices at maximal distance d from every vertex. This can be seen as a general version of the so-called spectral excess theorem, which allows us to characterize those distance-regular graphs which are half-antipodal, antipodal, bipartite, or with Kneser graph being strongly regular.

Keywords: Distance-regular graph, Kneser graph, Partial antipodality, Spectrum, Predistance polynomials.

1 Introduction

Let Γ be a distance-regular graph with adjacency matrix A and $d+1$ distinct eigenvalues. In the recent work of Brouwer and the author [1], we studied the situation where the distance-d graph Γ_d of Γ, or Kneser graph K, with adjacency matrix $A_d = p_d(A)$, has fewer distinct eigenvalues. In this case we say that Γ is *partially antipodal*. Examples are the so-called half antipodal (K with only one negative eigenvalue, up to multiplicity), and antipodal distance-regular graphs (K being disjoint copies of a complete graph). Here we generalize such a study to the case when Γ is a regular graph with $d+1$ distinct eigenvalues. The main result of this paper is a characterization of partially antipodal distance-regular graphs, among regular graphs with $d+1$ distinct eigenvalues, in terms of the spectrum and the mean number of vertices at maximal distance d from every vertex. This can be seen as a general version of the so-called spectral excess theorem, and allows us to characterize those distance-regular graphs which are half antipodal, antipodal, bipartite, or with Kneser graph being strongly regular.

Let Γ be a regular (connected) graph with degree k, n vertices, and spectrum $\operatorname{sp}\Gamma = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\}$, where $\lambda_0 (= k) > \lambda_1 > \cdots > \lambda_d$, and $m_0 = 1$. In this work, we use the so-called *predistance polynomials* $p_0, p_1, \dots, p_d$, introduced by the author and Garriga [5]. These are a sequence of orthogonal polynomials that can be seen as a generalization, for any graph, of the distance polynomials of distance-regular graphs. In fact, it is known that a regular graph Γ is distance-regular if and only if there exists a polynomial p of degree d such that $p(A) = A_d$, in which case $p = p_d$ (see [6]). Let Γ have diameter $D (\leq d)$. For $i = 0, \dots, D$, let $k_i(u)$ be the number of vertices at distance i from vertex u. Let $s_i(u) = k_0(u) + \cdots + k_i(u)$. In our work we use the following result, which can be seen as a version of the spectral excess theorem due to Garriga and the author [5] (for short proofs, see Van Dam [2], and Fiol, Gago and Garriga [4]):

Theorem 1.1 *Let Γ be a regular graph with spectrum $\operatorname{sp}\Gamma = \{\lambda_0, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\}$, where $\lambda_0 > \lambda_1 > \cdots > \lambda_d$. Let $\overline{s_i} = \frac{1}{n}\sum_{u \in V} s_i(u)$ be the average number of vertices at distance at most i from every vertex in Γ. Then, for any polynomial*

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$r \in \Re_{d-1}[x]$ *we have*

$$\frac{r(\lambda_0)^2}{\|r\|_\Gamma^2} \leq \overline{s_{d-1}}, \tag{1}$$

with equality if and only if Γ *is distance-regular and* r *is a nonzero multiple of* $q_{d-1} = p_0 + \cdots + q_{d-1}$.

2 The results

As mentioned above, in [1] we studied the situation where the distance-d graph Γ_d, of a distance-regular graph Γ with diameter d, has fewer distinct eigenvalues. Now, we are interested in the case when Γ is regular and with $d+1$ distinct eigenvalues. In this context, p_d is the highest degree predistance polynomial and, as $p_d(A)$ is not necessarily the distance-d matrix A_d (usually not even a 0-1 matrix), we consider the distinct eigenvalues of $p_d(A)$ vs. those of A. More precisely, given a set $H \subset \{0, \ldots, d\}$, we give conditions for all $p_d(\lambda_i)$ with $i \in H$ taking the same value. Notice that, because the values of p_d at the mesh $\lambda_0, \lambda_1, \ldots, \lambda_d$ alternate in sign, the feasible sets H must have either even or odd numbers

The case $\boldsymbol{\lambda_0 \notin H}$

We first study the more common case when $\lambda_0 \notin H$. For $i = 1, \ldots, d$, let $\phi_i(x) = \prod_{j\neq 0,i}(x - \lambda_i)$, and consider the Lagrange interpolating polynomial $L_i(x) = \phi_i(x)/\phi(\lambda_i)$, satisfying $L_i(\lambda_j) = \delta_{ij}$ for $j \neq 0$, and $L_i(\lambda_0) = (-1)^{i+1}(\pi_0/\pi_i)$, where $\pi_i = |\phi_i(\lambda_i)|$.

Theorem 2.1 *Let* Γ *be a regular graph with degree* k, n *vertices, and spectrum* $\operatorname{sp}\Gamma = \{\lambda_0, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, *where* $\lambda_0(= k) > \lambda_1 > \cdots > \lambda_d$. *Let* $H \subset \{1, \ldots, d\}$. *For every* $i = 0, \ldots, d$, *let* $\pi_i = \prod_{j\neq i} |\lambda_i - \lambda_j|$. *Let* $\overline{k_d} = \frac{1}{n}\sum_{u\in V} k_d(u)$ *be the average number of vertices at distance d from every vertex in* Γ. *Then,*

$$\overline{k_d} \leq \frac{n\sum_{i\in H} m_i}{\left(\sum_{i\in H} \frac{\pi_0}{\pi_i}\right)^2 + \sum_{i\notin H} \frac{\pi_0^2}{m_i \pi_i^2} \sum_{i\in H} m_i}, \tag{2}$$

and equality holds if and only if Γ *is a distance-regular graph with constant* $P_{id} = p_d(\lambda_i)$ *for every* $i \in H$.

As mentioned above, when Γ is already a distance-regular graph, Brouwer and the author [1] gave parameter conditions for partial antipodality, and surveyed known examples. The different examples given here are withdrawn from such a paper.

Example 2.2 The Odd graph O_5, on $n = 126$ vertices, has intersection array $\{5, 4, 4, 3; 1, 1, 2, 2\}$, so that $k_d = 60$, and spectrum $5^1, 3^{27}, 1^{42}, -2^{48}, -4^8$. Then, with $H = \{2, 4\}$, the function $\Phi(t)$ is depicted in Fig. 1. Its maximum is attained for $t_0 = 6$, and its value is $\Phi(6) = 66 = s_{d-1}$. Then, $P_{24} = P_{44}$.

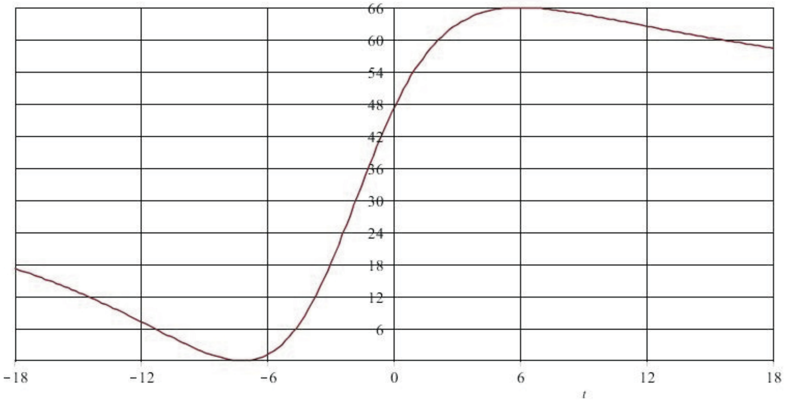

Fig. 1. The function $\Phi(t)$ for O_5 with $H = \{2, 4\}$.

Notice that if, in the above result, H is a singleton, there is no restriction for the values of p_d, and then we get the so-called spectral excess theorem [5].

Corollary 2.3 (The spectral excess theorem) *Let Γ be a regular graph with spectrum $\operatorname{sp}\Gamma$ and average number $\overline{k_d}$ as above. Then, Γ is distance-regular if and only if*

$$\overline{k_d} = p_d(\lambda_0) = n\left(\sum_{i=0}^{d} \frac{\pi_0^2}{m_i \pi_i^2}\right)^{-1}.$$

As mentioned before, in [1] a distance-regular graph Γ was said to be half antipodal if the distance-d graph has only one negative eigenvalue (i.e., P_{id} is a constant for every $i = 1, 3, \ldots$). Then, a direct consequence of Theorem 2.1 by taking $H = H_{\text{odd}} = \{1, 3, \ldots\}$ is the following characterization of half antipodality.

Corollary 2.4 *Let Γ be a regular graph as above. Then,*

(3) $$\overline{k_d} \leq \frac{n \sum_{i\ odd} m_i}{\left(\sum_{i\ odd} \frac{\pi_0}{\pi_i}\right)^2 + \sum_{i\ even} \frac{\pi_0^2}{m_i \pi_i^2} \sum_{i\ odd} m_i},$$

and equality holds if and only if Γ *is a half antipodal distance-regular graph.*

Recall that a regular graph is strongly regular if and only if it has at most three distinct eigenvalues (see e.g. [7]). Then, we can apply Theorem 2.1 with $H_{\text{even}} = \{2, 4, \ldots\}$ and $H_{\text{odd}} = \{1, 3, \ldots\}$ (and add up the two inequalities obtained) to obtain a characterization of those distance-regular graphs having strongly regular distance-d graph.

Corollary 2.5 *Let* Γ *be a regular graph as above. Then,*

$$\overline{k_d} \le \frac{n^2}{\left(\sum_{i\ even} \frac{\pi_0}{\pi_i}\right)^2 + \left(\sum_{i\ odd} \frac{\pi_0}{\pi_i}\right)^2 + \sum_{i\ even} \frac{\pi_0^2}{m_i \pi_i^2} \sum_{i\ odd} m_i + \sum_{i\ odd} \frac{\pi_0^2}{m_i \pi_i^2} \sum_{i\ even} m_i}, \tag{4}$$

and equality holds if and only if Γ *is a distance-regular graph with strongly regular distance-d graph* Γ_d*.*

Example 2.6 The Wells graph, on $n = 32$ vertices, has intersection array $\{5, 4, 1, 1; 1, 1, 4, 5\}$ and spectrum $5^1, \sqrt{5}^8, 1^{10}, -\sqrt{5}^8, -3^5$. This graph is 2-antipodal, so that $k_d = 1$. Then, Fig. 2 shows the functions $\Phi_0(t)$ with $H_0 = \{2, 4\}$, and $\Phi_1(t)$ with $H_1 = \{1, 3\}$. Their (common) maximum value is attained for $t_0 = 1$ and $t_1 = -1$, respectively, and it is $\Phi_0(1) = \Phi_1(-1) = 31 = s_{d-1}$. Then, $P_{24} = P_{44}$ and $P_{14} = P_{34}$.

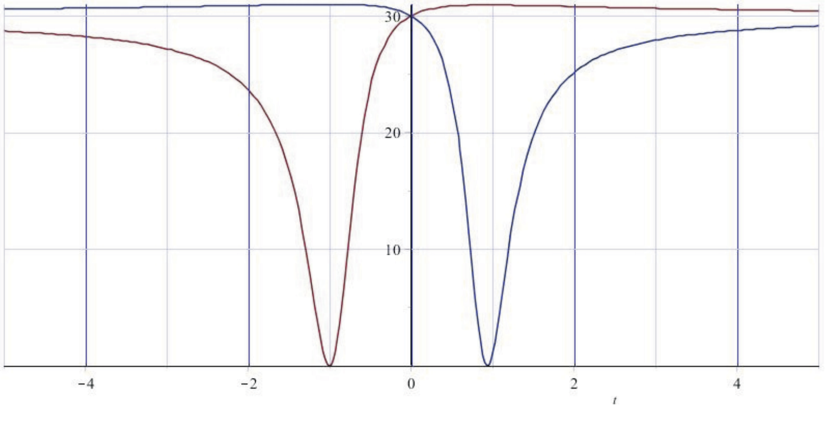

Fig. 2. The functions $\Phi_0(t)$ (in red) with $H_0 = \{2, 4\}$, and $\Phi_1(t)$ (in blue) with $H_1 = \{1, 3\}$ of the Wells graph.

The following result was used in [1,3] for the case of distance-regular graphs (where $p_d(\lambda_i) = P_{id}$).

Corollary 2.7 *Let Γ be a regular graph with eigenvalues $\lambda_0 > \lambda_1 > \cdots > \lambda_d$. Let $H \subset \{1, \ldots, d\}$. Then, $p_d(\lambda_i) = p_d(\lambda_j)$ for every $i, j \in H$ if and only if $\sum_{i \neq j}(m_i\pi_i - m_j\pi_j)^2 = 0$.*

The case $\boldsymbol{\lambda_0 \in H}$

To deal with this case, we use the following result which was proved in [1]:

Proposition 2.8 ([1, Prop. 8]) *Let Γ be a distance regular graph with diameter d. If $P_{0d} = P_{id}$ then i is even. Let $i > 0$ be even. Then $P_{0d} = P_{id}$ if and only Γ is antipodal, or $i = d$ and Γ is bipartite.*

Theorem 2.9 *Let Γ be a regular graph with n vertices, spectrum $\operatorname{sp}\Gamma$ as above, and mean excess $\overline{k_d}$. Then, for every $i = 1, \ldots, d$,*

$$\overline{k_d} \leq \frac{n\left(m_i + \sum_{j\neq 0,i} \frac{\pi_0^2}{m_j\pi_j^2}\right)}{\left(\frac{\pi_0}{\pi_i} + \sum_{j\neq 0,i} \frac{\pi_0^2}{m_j\pi_j^2}\right)^2 + m_i + \sum_{j\neq 0,i} \frac{\pi_0^2}{m_j\pi_j^2}}. \tag{5}$$

Moreover:

(*a*) *Equality holds for some $i \neq d$ if and only it holds for any $i = 1, \ldots, d$ and Γ is an antipodal distance-regular graph.*

(*b*) *Equality holds only for $i = d$ if and only if Γ is a bipartite, but not antipodal, distance-regular graph.*

Example 2.10 For the Wells graph the right hand expression of (5) gives $1(= k_4)$ for any $i = 1, \ldots, 4$, in concordance with its antipodal character. vertices, has intersection array $\{10, 9, 8, 7, 6; 1, 2, 3, 4, 10\}$ and spectrum 10^1,6^{45}, 2^{210}, -2^{210}, -6^{45}, -10^1. Then, the right hand expression of (5) gives 234.16, 293.36, 293.36, 234.16 for $i = 1, 2, 3, 4$, respectively, and $126(= k_5)$ for $i = 5$, showing that FQ_{10} is a bipartite distance-regular graph, but not antipodal.

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