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Jacobsthal numbers in generalised Petersen graphs

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Abstract

We prove that the number of 1-factorisations of a generalised Petersen graph of the type $GP(3k,k)$ is equal to the kth Jacobsthal number $J(k)$ if k is odd, and equal to $4J(k)$, when k is even. Moreover, we verify the list colouring conjecture for $GP(3k,k)$.

Keywords: Jacobsthal numbers, generalized Petersen graphs, list colouring conjecture, 1-factorizations, edge-colouring

1 Introduction

Often, combinatorial objects that on the surface seem quite different nevertheless exhibit a deeper, somewhat hidden, connection. This is, for instance, the case for tilings of $3\times(k-1)$-rectangles with 1×1 and 2×2-squares [12], certain meets in lattices [7], and the number of walks of length k between adjacent vertices in a triangle [3]: in all three cases the cardinality is equal to the

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kth *Jacobsthal number*. Their sequence $0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341\ldots$ is defined by the recurrence relation $J(k) = J(k-1) + 2J(k-2)$ and initial values $J(0) = 0$ and $J(1) = 1$. Jacobsthal numbers also appear in the context of alternating sign matrices [10], the Collatz problem and in the study of necktie knots [9]; see [13, A001045] for much more.

We add to this list by describing a relationship to certain *generalised Petersen graphs* $GP(3k, k)$. Generalised Petersen graphs were first studied by Coxeter [6]. For $k, n \in \mathbb{N}$ with $k < \frac{n}{2}$, the graph $GP(n, k)$ is defined as the graph on vertex set $\{u_i, v_i : i \in \mathbb{Z}_n\}$ with edge set $\{u_iu_{i+1}, u_iv_i, v_iv_{i+k} : i \in \mathbb{Z}_n\}$. See Figure 1 for two examples.

Theorem 1.1 *For odd k, the number of 1-factorisations of the generalised Petersen graph $GP(3k, k)$ equals the Jacobsthal number $J(k)$; for even k, the number is equal to $4J(k)$.*

A 1-*factorisation* of a graph $G = (V, E)$ is a partition of the edge set into perfect matchings. (A *perfect matching* is a set of $^{|V|}/_2$ edges, no two of which share an endvertex.) Such factorisations are closely linked to edge colourings: indeed, a d-regular graph G has a 1-factorisation if and only if its edge set can be coloured with d colours. That is, the *chromatic index*, the minimal number of colours needed to colour all the edges, is equal to d.

List edge-colourings generalise edge colourings. Given lists L_e of allowed colours at every edge $e \in E$, the task consists in colouring the edges so that every edge e receives a colour from its list L_e. The *choice index* of G is the smallest number ℓ so that any collection of lists L_e of size ℓ each allows a list colouring. The choice index is at least as large as the chromatic index. The famous *list-colouring conjecture* asserts that the two indices never differ:

Conjecture 1.2 (List-colouring conjecture) *The chromatic index of every simple graph equals its choice index.*

While the conjecture has been verified for some graph classes, bipartite graphs [11] and regular planar graphs [8] for instance, the conjecture remains wide open for most graph classes, among them cubic graphs. We prove:

Theorem 1.3 *The list-colouring conjecture is true for generalised Petersen graphs $GP(3k, k)$.*

Generalised Petersen graphs are cubic graphs. All of them, except the Petersen graph itself, have chromatic index 3; see Watkins [14], and Castagna and Prins [5]. In particular, this means that the list colouring conjecture for them does not follow from the list version of Brooks' theorem.

Our proof is based on the algebraic colouring criterion of Alon and Tarsi [2]. In our setting, it suffices to check that, for a suitable definition of a sign, the number of positive 1-factorisations differs from the number of negative 1-factorisations. In this respect our second topic ties in quite nicely with our first, and we will be able to re-use some of the observations leading to Theorem 1.1.

The following two sections provide the main ideas to prove Theorem 1.1 and 1.3. All details can be found in the full paper [4].

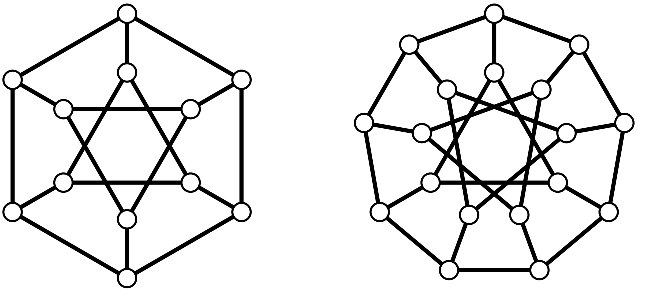

Fig. 1. The Dürer graph $GP(6,2)$ and the generalised Petersen graph $GP(9,3)$

2 Counting 1-factorisations

We consider a fixed generalised Petersen graph $GP = GP(3k,k)$. The *outer cycle* C_O of GP, the cycle $u_1u_2\dots u_{3k-1}u_{3k}u_1$ plays a key role.

Our objective is to count the number of 1-factorisations of GP. Rather than counting them directly, we consider edge colourings and show that it suffices to focus on certain edge colourings of the outer cycle.

Let ϕ be an edge colouring with colours $\{1,2,3\}$ of either the whole graph GP or only of the outer cycle C_O. We split ϕ into k triples

$$\phi_i = \left(\phi\left(u_iu_{i+1}\right), \phi\left(u_{k+i}u_{k+i+1}\right), \phi\left(u_{2k+i}u_{2k+i+1}\right)\right) \text{ for } i = 1,\dots,k.$$

To keep notation simple, we will omit the parentheses and commas, and only write $\phi_i = 123$ to mean $\phi_i = (1,2,3)$. We, furthermore, define also $\phi_{k+1} = \left(\phi\left(u_{k+1}u_{k+2}\right), \phi\left(u_{2k+1}u_{2k+2}\right), \phi\left(u_1u_2\right)\right)$, and note that ϕ_{k+1} is obtained from ϕ_1 by a cyclic shift.

It turns out that the colours on the outer cycle already uniquely determine the edge colouring on the whole graph. Moreover, it is easy to describe which colourings of the outer cycle extend to the rest of the graph:

Lemma 2.1 *Let $\phi : E(C_O) \to \{1, 2, 3\}$ be an edge colouring of C_O. Then the following two statements are equivalent:*

(i) *there is an edge colouring γ of GP with $\gamma|_{C_O} = \phi$; and*

(ii) *there is a permutation $(\mathsf{a}, \mathsf{b}, \mathsf{c})$ of $(1, 2, 3)$ so that ϕ_i and ϕ_{i+1} are for all $i = 1, \ldots, k$ adjacent vertices in one of the graphs T and H in Figure 2.*

Furthermore, if there is an edge colouring γ of GP as in (i) then it is unique.

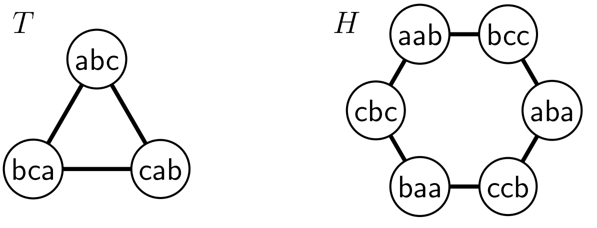

Fig. 2. The graphs T and H capture the possible combinations of consecutive colour triples

The lemma implies that any edge colouring γ of GP corresponds to a walk $\gamma_1\gamma_2 \ldots \gamma_{k+1}$ of length k in either T or in H. Where does such a walk start and end? By symmetry, we may assume that the walk starts at $\gamma_1 = \mathsf{abc}$ or $\gamma_1 = \mathsf{aab}$. It then ends in γ_{k+1}, which is either bca or aba. Conversely, all such walks define edge colourings of GP.

To count the number of these walks, consider two vertices x, y of T, resp. of H, that are at distance ℓ from each other in T (resp. in H). We define

$$t_k(\ell) := \sharp\,\{\text{walks of length } k \text{ between } x \text{ and } y \text{ in } T\}$$
$$h_k(\ell) := \sharp\,\{\text{walks of length } k \text{ between } x \text{ and } y \text{ in } H\}$$

Then every edge colouring of GP corresponds to a walk that is either counted in $t_k(1)$ (as abc and bca have distance 1 in T) or counted in $h_k(2)$.

Lemma 2.2 *The number of 1-factorisations of $GP(3k, k)$ equals $t_k(1)+3h_k(2)$.*

Since H covers T, the walks counted by $h_k(2)$ and $t_k(1)$ are in one-to-one correspondence and we obtain

Lemma 2.3 *$h_k(2)$ equals $t_k(1)$ for even k.*

By [3], $t_k(1)$ equals $J(k)$ which concludes the proof of Theorem 1.1.

3 List edge colouring

In order to show the list edge conjecture for $GP = GP(3k, k)$, we use the method of Alon and Tarsi [2], or rather its specialisation to regular graphs [8].

To define a local rotation, we consider $GP(3k, k)$ always to be drawn as in Figure 1: the vertices u_i for $i = 1, \ldots, 3k$ are placed on an outer circle in clockwise order, the vertices v_i for $i = 1, \ldots, 3k$ on a smaller concentric circle in such a way that u_i and v_i match up, and all edges are straight. We define the sign of γ at a vertex w as $+$ if the colours $\mathsf{1}, \mathsf{2}, \mathsf{3}$ appear in clockwise order on the incident edges; otherwise the sign is $-$. The sign of the colouring γ is then

$$\operatorname{sgn}(\gamma) := \prod_{v \in V(GP)} \operatorname{sgn}_\gamma(v).$$

Since our graphs are regular, permuting colours in our context does not change the sign of an edge colouring, see e.g. [8]. This allows to define a sign $\operatorname{sgn} f$ for any 1-factorisation f by fixing it to the sign of any edge colouring that induces f. The Alon-Tarsi colouring criterion now takes a particularly simple form in d-regular graphs; see Ellingham and Goddyn [8] or Alon [1]: A d-regular graph is d-list-edge-colourable if $\sum_{f \text{ 1-factor of } G} \operatorname{sgn}(f) \neq 0$.

Since the number of 1-factorisations of $GP(3k, k)$ with odd k is the odd number $J(k)$, application of the criterion yields:

Corollary 3.1 *For odd k, the graph $GP(3k, k)$ has choice index* 3.

Unfortunately, for even k the number of 1-factorisations is even. That means, we need to count the positive and negative 1-factorisations separately.

As a first step, we refine the colour triple graphs T and H, and endow them with signs on the edges. Figure 3 shows the graphs $T_\pm$ and $H_\pm$, which we obtain from T and H by replacing each edge by two inverse directed edges, each having a sign. Note that in $T_\pm$ all edges in clockwise direction are positive, while clockwise edges in $H_\pm$ are negative.

For two adjacent vertices x, y in $T_\pm$ or in $H_\pm$ we denote the *sign of the edge* pointing from x to y by $\operatorname{sgn}(x, y)$. The next lemma shows that the signs on the edges capture the signs of edge colourings.

Lemma 3.2 *Let $\gamma : E(GP) \to \{\mathsf{1}, \mathsf{2}, \mathsf{3}\}$ be an edge colouring of GP, and let $(\mathsf{a}, \mathsf{b}, \mathsf{c})$ be a permutation of $(\mathsf{1}, \mathsf{2}, \mathsf{3})$ so that γ_1 is a vertex in $T_\pm$ or in $H_\pm$. Then*

$$\operatorname{sgn}(\gamma) = \prod_{i=1}^{k} \operatorname{sgn}(\gamma_i, \gamma_{i+1}).$$

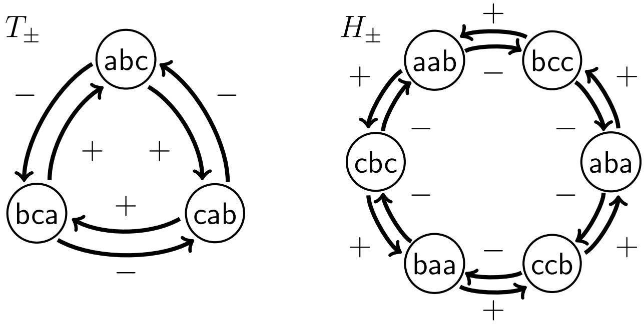

Fig. 3. Signs of the possible combinations of consecutive colour triples

Lemmas 2.1 and 3.2 imply that every positive 1-factorisation corresponds to a walk in either $T_\pm$ or $H_\pm$ whose edge signs multiply to $+$. We call such a walk *positive*; whereas a walk whose signs multiply to $-$ is *negative*.

To count such walks, we observe that not only the distance between two vertices has an influence on the sign of a walk between them, but also the rotational direction of the shortest path.

For two vertices x and y for which the clockwise path from x to y is of length ℓ, we define

$$t_k^+(\ell) := \sharp\,\{\text{positive walks of length } k \text{ from } x \text{ to } y \text{ in } T_\pm\}$$
$$h_k^+(\ell) := \sharp\,\{\text{positive walks of length } k \text{ from } x \text{ to } y \text{ in } H_\pm\}$$

and $t_k^-(\ell)$ and $h_k^-(\ell)$ analogously.

Similarly as in Section 2 for unsigned colourings, every positive edge colouring of GP now corresponds to a positive walk in $T_\pm$ or in $H_\pm$. Since all edge colourings with the same associated 1-factorisation have the same sign, we thus have a way to count positive and negative 1-factorisations via walks in signed graphs:

Lemma 3.3 *The number of positive/negative* 1*-factorisations of* $GP(3k, k)$ *equals* $t_k^\pm(2) + 3h_k^\pm(2)$.

As in Lemma 2.3 we can state a connection between walks in $T_\pm$ and $H_\pm$.

Lemma 3.4 $h_k^\pm(2)$ *equals* $t_k^\pm(2)$ *for even* k.

Using recurrence relations between positive and negative walks with different endpoints yields for any integer $k \geq 1$

$$t_k^+(2) = \frac{1}{6}\left(2^k - (-1)^k\left(1 + (-3)^{\lceil \frac{k}{2} \rceil}\right)\right) \quad t_k^-(2) = \frac{1}{6}\left(2^k - (-1)^k\left(1 - (-3)^{\lceil \frac{k}{2} \rceil}\right)\right). \tag{1}$$

This shows that the number $4t_k^+(2)$ of positive 1-factorisations of $GP(3k,k)$ with even k does not equal the number $4t_k^-(2)$ of negative ones. Together with Corollary 3.1, this proves Theorem 1.3.

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