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Bounding the Clique-Width of H-free Split Graphs

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Abstract

A graph is H-free if it has no induced subgraph isomorphic to H. We continue a study into the boundedness of clique-width of subclasses of perfect graphs. We identify five new classes of H-free split graphs whose clique-width is bounded. Our main result, obtained by combining new and known results, provides a classification of all but two stubborn cases, that is, with two potential exceptions we determine *all* graphs H for which the class of H-free split graphs has bounded clique-width.

Keywords: clique-width, split graphs, perfect graphs, forbidden induced subgraph, hereditary graph class

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1 Introduction

The *clique-width* of a graph G, denoted $\mathrm{cw}(G)$, is the minimum number of labels needed to construct G by using the following four operations:

(i) creating a new graph consisting of a single vertex v with label i;

(ii) taking the disjoint union of two labelled graphs G_1 and G_2;

(iii) joining each vertex with label i to each vertex with label j $(i \neq j)$;

(iv) renaming label i to j.

Clique-width is a well-studied graph parameter; see for example the surveys of Gurski [10] and Kamiński, Lozin and Milanič [12]. A graph class is said to be of *bounded* clique-width if there is a constant p such that the clique-width of every graph in the class is at most p. Much research has been done identifying whether or not various classes have bounded clique-width. For instance, the Information System on Graph Classes and their Inclusions [8] maintains a record of graph classes for which this is known. In a recent series of papers [2,5,6,7] the clique-width of graph classes characterized by two forbidden induced subgraphs was investigated. In particular we refer to [7] for details on how new results can be combined with known results to give a classification for all but 13 open cases (up to an equivalence relation). Similar studies have been performed for variants of clique-width, such as linear clique-width [11] and power-bounded clique-width [1]. Moreover, the (un)boundedness of the clique-width of a graph class seems to be related to the computational complexity of the Graph Isomorphism problem, which has in particular been investigated for graph classes defined by two forbidden induced subgraphs [13,16].

In this paper we continue a study into the boundedness of the clique-width of subclasses of perfect graphs. Clique-width is still one of the most difficult graph parameters to deal with. For instance, deciding whether or not a graph has clique-width at most c for some fixed constant c is only known to be polynomial-time solvable if $c \leq 3$ [3], but is a long-standing open problem for $c \geq 4$. Our long-term goal is to increase our understanding of clique-width. To this end we aim to identify new classes of bounded clique-width. In order to explain some previously known results, along with our new ones, we first give some terminology.

Terminology. For two vertex-disjoint graphs G and H, the *disjoint union* $(V(G) \cup V(H), E(G) \cup E(H))$ is denoted by $G + H$ and the disjoint union of r copies of G is denoted by rG. The *complement* of a graph G, denoted by $\overline{G}$, has vertex set $V(\overline{G}) = V(G)$ and an edge between two distinct vertices if and

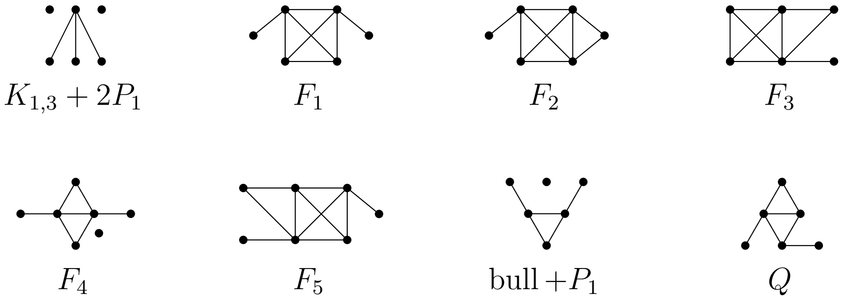

Fig. 1. The graphs $K_{1,3}+2P_1, F_1, F_2, F_3, F_4, F_5$, bull $+P_1$ and Q from Theorems 1.2 and 1.4.

only if these vertices are not adjacent in G. For two graphs G and H we write $H \subseteq_i G$ to indicate that H is an induced subgraph of G. A set M of vertices is a *module* if every vertex not in M is either adjacent to every vertex of M or non-adjacent to every vertex of M. A module of G is *trivial* if it contains zero, one or all vertices of G. A graph G is *prime* if every module in G is trivial. The graphs $C_r, K_r, K_{1,r-1}$ and P_r denote the cycle, complete graph, star and path on r vertices, respectively. The graph $S_{h,i,j}$, for $1 \leq h \leq i \leq j$, denotes the *subdivided claw*, that is the tree that has only one vertex x of degree 3 and exactly three leaves, which are of distance h, i and j from x, respectively. For a set of graphs $\{H_1,\ldots,H_p\}$, a graph G is $(H_1,\ldots,H_p)$*-free* if it has no induced subgraph isomorphic to a graph in $\{H_1,\ldots,H_p\}$. The *bull* is the graph with vertices a,b,c,d,e and edges ab,bc,ca,ad,be. A graph G is *chordal* if it is $(C_4,C_5,\ldots)$-free and *weakly chordal* if both G and $\overline{G}$ are $(C_5,C_6,\ldots)$-free. Every chordal graph is weakly chordal and every weakly chordal graph is perfect.

Known Results on Subclasses of Perfect Graphs. We start off with the following known theorem, which shows that the restriction of H-free graphs to H-free weakly chordal graphs does not yield any new graph classes of bounded clique-width, as both classifications are exactly the same.

Theorem 1.1 ([2,7]) *Let H be a graph. The class of H-free (weakly chordal) graphs has bounded clique-width if and only if H is an induced subgraph of P_4.*

Motivated by Theorem 1.1 we investigated classes of H-free chordal graphs in an attempt to identify *new* classes of bounded clique-width and as a (successful) means to find reductions to solve more cases in our classification for (H_1,H_2)-free graphs. This classification for classes of H-free chordal graphs is almost complete except for two cases (see also Fig. 1).

Theorem 1.2 ([2]) *Let H be a graph not in $\{F_1,F_2\}$. The class of H-free* *chordal graphs has bounded clique-width if and only if*

- $H = K_r$ *for some* $r \geq 1$*;*
- $H \subseteq_i$ bull*;*
- $H \subseteq_i P_1 + P_4$*;*
- $H \subseteq_i \overline{P_1 + P_4}$*;*
- $H \subseteq_i \overline{K_{1,3} + 2P_1}$*;*
- $H \subseteq_i P_1 + \overline{P_1 + P_3}$*;*
- $H \subseteq_i P_1 + \overline{2P_1 + P_2}$ *or*
- $H \subseteq_i \overline{S_{1,1,2}}$.

In contrast to chordal graphs, the classification for bipartite graphs, another class of perfect graphs, is complete. This classification was used in the proof of Theorem 1.2 and it is similar to a characterization of Lozin and Volz [14] for a different variant of the notion of H-freeness in bipartite graphs (see [6] for an explanation of the difference).

Theorem 1.3 ([6]) *Let H be a graph. The class of H-free bipartite graphs has bounded clique-width if and only if*

- $H = sP_1$ *for some* $s \geq 1$*;*
- $H \subseteq_i K_{1,3} + 3P_1$*;*
- $H \subseteq_i K_{1,3} + P_2$*;*
- $H \subseteq_i P_1 + S_{1,1,3}$ *or*
- $H \subseteq_i S_{1,2,3}$.

Our Results. We consider subclasses of split graphs. A graph $G = (V, E)$ is a *split graph* if it has a *split partition*, that is, a partition of V into two (possibly empty) sets K and I, where K is a clique and I is an independent set. The class of split graphs coincides with the class of $(2K_2, C_4, C_5)$-free graphs [9] and is known to have unbounded clique-width [15]. As with the previous graph classes we forbid one additional induced subgraph H. We aim to classify the boundedness of clique-width for H-free split graphs and to identify new graph classes of bounded clique-width along the way. Theorem 1.2 also provides motivation, as it would be useful to know whether the clique-width of H-free split graphs is bounded when $H = F_1$ or $H = F_2$ (the two missing cases for chordal graphs). We give affirmative answers for both of these cases. It should be noted that the complement of a split graph is split and that complementation preserves boundedness of clique-width [12]. Hence, for any graph H, the class of H-free split graphs has bounded clique-width if and only if the class of $\overline{H}$-free split graphs has bounded clique-width. As such our main result shows that there are only two open cases.

Theorem 1.4 *Let H be a graph such that neither H nor $\overline{H}$ is in $\{F_4, F_5\}$. The class of H-free split graphs has bounded clique-width if and only if*

- *H or $\overline{H} = rP_1$ for some $r \geq 1$;*
- *H or $\overline{H} \subseteq_i \text{bull} + P_1$;*
- *H or $\overline{H} \subseteq_i F_1$;*
- *H or $\overline{H} \subseteq_i F_2$;*
- *H or $\overline{H} \subseteq_i F_3$;*
- *H or $\overline{H} \subseteq_i Q$ or*
- *H or $\overline{H} \subseteq_i K_{1,3} + 2P_1$.*

2 Our Techniques

A *labelled* bipartite graph $H^\ell = (B_H^\ell, W_H^\ell, E_H)$ consists of a bipartite graph H together with a labelling ℓ that assigns either the colour "black" or the colour "white" to each vertex of H in such a way that the two resulting monochromatic colour classes B_H^ℓ and W_H^ℓ form a partition of H into two (possibly empty) independent sets. Note that the triples $(B_H^\ell, W_H^\ell, E_H)$ and $(W_H^\ell, B_H^\ell, E_H)$ correspond to different labelled bipartite graphs. Two labelled bipartite graphs H_1^ℓ and $H_2^{\ell^*}$ are *isomorphic* if the (unlabelled) graphs H_1 and H_2 are isomorphic, and if in addition there exists an isomorphism $f : V(H_1) \to V(H_2)$ such that for all $u \in V(H_1)$, we have that $u \in W_{H_1}^\ell$ if and only if $f(u) \in W_{H_2}^{\ell^*}$. We write $H_1^\ell \subseteq_{li} H_2^{\ell^*}$ if $H_1 \subseteq_i H_2$, $B_{H_1}^\ell \subseteq B_{H_2}^{\ell^*}$ and $W_{H_1}^\ell \subseteq W_{H_2}^{\ell^*}$, in which case we say that H_1^ℓ is a *labelled* induced subgraph of $H_2^{\ell^*}$. An (unlabelled) bipartite graph G is *weakly* H^ℓ-free if there is a labelling ℓ^* of G such that G^{ℓ^*} does not contain H^ℓ as a labelled induced subgraph. If ℓ is a labelling of H, we let $\overline{\ell}$ denote the *opposite* labelling of H, in which the colours are swapped. If a bipartite graph H has a unique (up to isomorphism) labelling in which the number of black vertices is maximised, we call this labelling b. We will make use of the following theorem (see also Fig. 2).

Theorem 2.1 ([6]) *Let H^ℓ be a labelled bipartite graph. The class of weakly H^ℓ-free bipartite graphs has bounded clique-width if and only if one of the following cases holds:*

- *H^ℓ or $H^{\overline{\ell}} = (sP_1)^b$ for some $s \geq 1$;*

- *H^{ℓ} or $H^{\overline{\ell}} \subseteq_{li} (P_1 + P_5)^b$;*
- *$H^{\ell} \subseteq_{li} (P_2 + P_4)^b$ or*
- *$H^{\ell} \subseteq_{li} (P_6)^b$.*

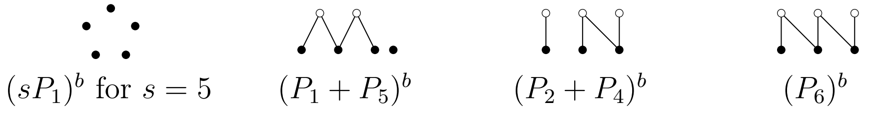

Fig. 2. The labelled bipartite graphs from Theorem 2.1.

Similarly to the way that a bipartite graph can have multiple labellings, a split graph G may have multiple split partitions, say (K_1, I_1) and (K_2, I_2). We say that two such split partitions are *isomorphic* if there is an isomorphism $f : V(G) \to V(G)$ of G such that $u \in K_1$ if and only if $f(u) \in K_2$. By exploring the properties of split partitions and using Theorem 2.1, we are able to prove the following:

Lemma 2.2 *If the class of H-free split graphs has bounded clique-width then H or $\overline{H}$ is isomorphic to K_r for some r or is an induced subgraph of F_4 or F_5.*

Note that both F_4 and F_5 have seven vertices. The six-vertex induced subgraphs of F_4 are: bull $+P_1$, $\overline{F_1}$, $\overline{F_3}$ and $K_{1,3} + 2P_1$. The six-vertex induced subgraphs of F_5 are: bull $+P_1$, F_1, F_2, $\overline{F_2}$, F_3, $\overline{F_3}$ and Q. These graphs and their complements are precisely the cases listed in Theorem 1.4. The $H = rP_1$, bull $+P_1$ and Q cases of Theorem 1.4 follow from Theorem 2.1 using similar arguments to those used to prove Lemma 2.2. The $H = K_{1,3} + 2P_1$ case follows from Theorem 1.2. To prove the $H = F_1, F_2$ and F_3 cases we make use of Theorem 2.1 combined with the following lemma, which allows us to restrict ourselves to studying prime graphs in the respective classes.

Lemma 2.3 ([4]) *If $\mathcal{P}$ is the set of all prime induced subgraphs of a graph G then* $\mathrm{cw}(G) = \max_{H \in \mathcal{P}} \mathrm{cw}(H)$.

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