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The Circuit Diameter of the Klee-Walkup Polyhedron

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Abstract

Consider a variant of the graph diameter of a polyhedron where each step in a walk between two vertices travels maximally in a circuit direction instead of along incident edges. Here circuit directions are non-trivial solutions to minimally-dependent subsystems of the presentation of the polyhedron. These can be understood as the set of all possible edge directions, including edges that may arise from translation of the facets.

It is appealing to consider a circuit analogue of the Hirsch conjecture for graph diameter, as suggested by Borgwardt et al. [2]. They ask whether the known counterexamples to the Hirsch conjecture give rise to counterexamples for this relaxed notion of circuit diameter. We show that the most basic counterexample to the unbounded Hirsch conjecture, the Klee-Walkup polyhedron, does have a circuit diameter that satisfies the Hirsch bound, regardless of representation.

Keywords: Circuit diameter, Hirsch conjecture, Klee-Walkup polyhedron

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1 Introduction

Given a convex polyhedron, its *graph* (also called its *skeleton*) is the undirected graph formed by its vertices and edges. The *graph diameter* of a polyhedron is then defined to be the graph diameter of its skeleton. This is an interesting quantity since it gives us a lower bound on the performance of the simplex method for linear programming. It turns out that most known polyhedra have diameters at most the *Hirsch bound* of $f-d$, where f and d are the number of facets and the dimension, respectively. The main exceptions are unbounded polyhedra based on the Klee-Walkup example [3] and non-Hirsch polytopes based on the constructions of Santos [4].

Here we consider the circuit diameter, where instead of being restricted to walk along edges of a polyhedron, one can walk in the direction of any 'potential edges' obtained by translating its facets. In particular, the set of permissible directions at a vertex has no connection with edges incident to it; circuit steps can go through the interior of the polyhedron and end when the direction is traversed as far as possible maintaining feasibility. Because of this, it is challenging to prove anything about the circuit diameter independent of how the polyhedron is realized. In this note, we show that the original unbounded non-Hirsch polyhedron of Klee and Walkup [3] does satisfy the Hirsch bound in this relaxed framework, independent of realization.

2 Circuit Walks and Diameters

2.1 Background and Definitions

While the graph diameter of a polyhedron considers walks along its edges, the circuit diameter considers walks that use the *circuits* of a polyhedron, defined as follows:

Definition 2.1 Given a polyhedron

$$P = \{x \in \mathbb{R}^n : A^1 x = b^1, A^2 x \geq b^2\},$$

where $A^i \in \mathbb{Q}^{d_i \times n}$ and $b^i \in \mathbb{R}^{d_i}$ for $i = 1, 2$, the *circuits* or *elementary vectors* $\mathcal{C}(A^1, A^2)$ of A^1 and A^2 are defined as the set of vectors $g \in \ker(A^1) \setminus \{0\}$ for which $A^2 g$ is support-minimal in the set $\{A^2 x : x \in \ker(A^1) \setminus \{0\}\}$, where g is normalized to coprime integer components.

It turns out that the set $\mathcal{C}(A^1, A^2)$ consists of exactly the possible edge directions of P for varying b^1 and b^2 [7]. Moreover, at any non-optimal feasible point of the linear program

$$\min\{c^T x : A^1 x = b^1, A^2 x \geq b^2\},$$

an augmenting direction can always be found from the set $\mathcal{C}(A^1, A^2)$.

Now, for a polyhedron P and the set of circuits $\mathcal{C}$ associated with A^1 and A^2, given two vertices u and v of P define a *circuit walk of length k* to be a sequence $u = y^0, \ldots, y^k = v$ with

(i) $y^i \in P$

(ii) $y^{i+1} - y^i = \alpha_i g^i$ for some $g^i \in \mathcal{C}$ and $\alpha_i > 0$

(iii) $y^i + \alpha g_i \notin P$ for $\alpha > \alpha_i$

for all $i = 0, 1, \ldots, k-1$. Observe that edge walks from u to v are exactly those circuit walks where each pair y^i, y^{i+1} are adjacent vertices of P. Hence by considering all possible circuit directions at each point y^i, instead of directions corresponding to incident edges, we see that the circuit walks are generalizations of edge walks. The *circuit distance* from u to v is now defined as the length of the shortest circuit walk from u to v, and the *circuit diameter* of P is the largest circuit distance between any two vertices of P. Observe that the circuit distance is a lower bound for the graph distance. For more context on circuit diameter, see [2] and [1].

2.2 Variants of the Hirsch Conjecture

There are polyhedra whose circuit diameter and graph diameter are the same – a trivial example is the d-dimensional simplex, which has both graph and circuit diameter equal to 1. Also, take the d-dimensional cube with vertices all vectors in $\{0,1\}^d$. Its facet description is

$$\{(x_1, x_2, \ldots, x_d) : 0 \leq x_i \leq 1, i = 1, 2, \ldots, d\}.$$

This representation of the d-cube has circuits $\{\pm e_1, \pm e_2, \ldots, \pm e_d\}$, where e_i is the vector with a 1 in the ith position and 0's elsewhere. Hence its circuit diameter is also d. It is not clear if this can change in another representation of the d-cube.

The simplex and the cube are critical examples that motivated the well-known Hirsch conjecture. The conjecture can be stated as follows:

Conjecture 2.2 (Hirsch, 1957) *Let $f > d \geq 2$. Let P be a d-dimensional polyhedron with f facets. Then the combinatorial diameter of P is at most $f - d$.*

This is not true in general. Klee and Walkup in [3] found a counterexample that is an unbounded polyhedron in dimension 4, with 8 facets and diameter 5; this is featured in the next section. The bounded case was finally settled by Santos [4], which has stimulated activity in this area. The conjecture does however hold for many interesting classes of polyhedra. See [5] for a survey of recent research related to the Hirsch conjecture.

While the the Hirsch conjecture admits some hard-to-find exceptions for graph diameter, the situation for circuit diameter is still unresolved.

Conjecture 2.3 *[2] The circuit diameter of a d-dimensional polyhedron with f facets is bounded above by $f - d$.*

In Section 3, we show that for circuit diameter, the most basic non-Hirsch unbounded polyhedron actually does satisfy the Hirsch bound, independent of representation. This provides some evidence for Conjecture 2.3 by establishing it in one place where Hirsch does not hold for the graph diameter.

3 The Klee-Walkup Polyhedron

The first unbounded non-Hirsch polyhedron was given by Klee and Walkup in [3], where they constructed a 4-dimensional polyhedron with 8 facets and diameter 5. Its facet description is $\{x \in \mathbb{R}^4 : Ax \geq b\}$ where

$$A = \begin{pmatrix} -6 & -3 & 0 & 1 \\ -3 & -6 & 1 & 0 \\ -35 & -45 & 6 & 3 \\ -45 & -35 & 3 & 6 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} -1 \\ -1 \\ -8 \\ -8 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We call the combinatorial class of this polyhedron U_4, and this particular realization by $\tilde{U}_4$. Its vertex-edge graph is shown in Figure 1 [5]. Here the

[5] All illustrations in this note were produced using the Geogebra software, see `http://www.geogebra.org`.

vertices are indexed by the four facets containing each one, while the points labelled with `R`'s represent extreme rays. It is clear from the graph that vertices `V5678` and `V1234` are at graph distance five apart.

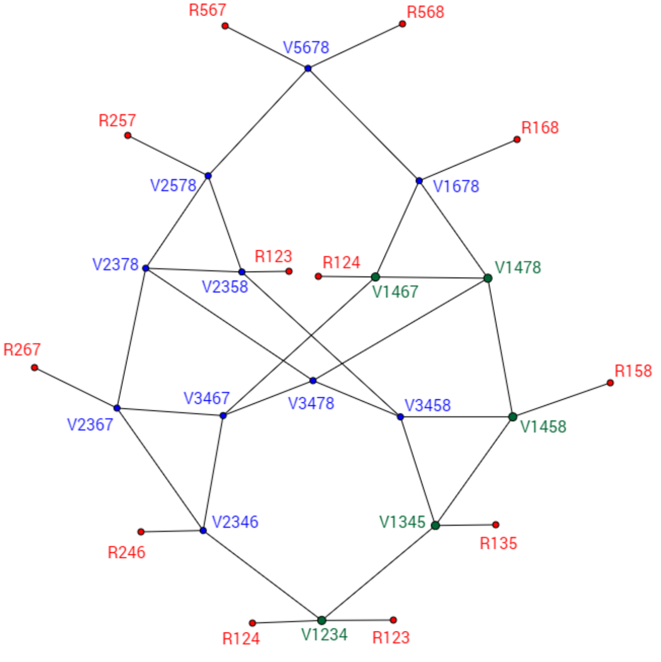

Fig. 1. The skeleton of U_4.

We now prove the following result:

Theorem 3.1 *The circuit diameter of the Klee-Walkup polyhedron U_4 is at most 4, independent of representation.*

Proof. First we demonstrate the existence of a circuit walk of length 4 from `V5678` to `V1234`. Observe that we can take two edge steps as follows: `V5678` $\rightarrow$ `V1678` $\rightarrow$ `V1478`. Vertices `V1478` and `V1234` are both contained in the 2-face determined by facets `1` and `4`, so we can complete the walk on this face. Note that this 2-face is an unbounded polyhedron on six facets. Figure 2 is a topological illustration of this face, showing the order of the vertices and rays.

Now consider a vector $\boldsymbol{g}$ corresponding to the edge direction from `V1458` to `V1345` – this is the blue vector in Figure 3. Note that this is always a circuit direction in any representation of U_4 since it corresponds to an actual edge of the polyhedron.

To see that $\boldsymbol{g}$ is a feasible direction at `V1478`, consider vector $\boldsymbol{h}$ in the edge direction from `V1478` to `V1458`, and vector $\boldsymbol{r}$ in the direction of ray `R124`.

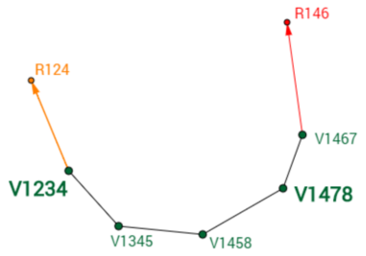

Fig. 2. The 2-face determined by facets `1` and `4`.

Observe that $\boldsymbol{g}$ and $-\boldsymbol{h}$ are the two incident edge directions at `V1458`, and so $\boldsymbol{r}$ must be a strict conic combination of $\boldsymbol{g}$ and $-\boldsymbol{h}$, i.e. $\boldsymbol{r} = \alpha_1(\boldsymbol{g}) + \alpha_2(-\boldsymbol{h})$ for $\alpha_1, \alpha_2 > 0$. By rearranging terms we see that $\boldsymbol{g}$ is a strict conic combination of $\boldsymbol{h}$ and $\boldsymbol{r}$: $\boldsymbol{g} = (\alpha_2/\alpha_1)\boldsymbol{h} + (1/\alpha_1)\boldsymbol{r}$, with $\alpha_2/\alpha_1, 1/\alpha_1 > 0$. Feasibility of $\boldsymbol{r}$ and $\boldsymbol{h}$ at `V1478` implies that $\boldsymbol{g}$ is a feasible direction at `V1478`.

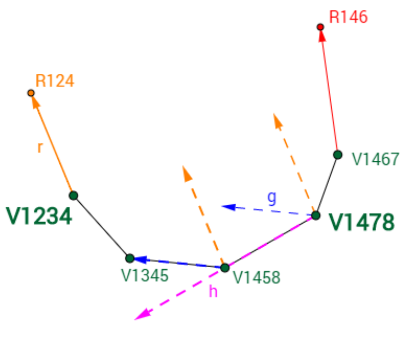

Fig. 3. Feasibility of the circuit direction $\boldsymbol{g}$.

Now starting at `V1478` traverse $\boldsymbol{g}$ as far as feasibility allows. This direction is bounded since we exit the polyhedron when taking $\boldsymbol{g}$ from `V1458`. We will eventually exit the 2-face at a point along the boundary, and at one of the following positions:

- exactly at `V1234`,
- on the edge connecting `V1234` and `V1345`, or
- on the ray `R124` emanating from `V1234`.

Hitting exactly `V1234` gives a circuit walk of length 3 from `V5678`, while the other two cases give circuit walks of length 4 since we only need one step to `V1234`. These two situations are illustrated in Figure 4.

The argument is the same for the reverse direction (`V1234` to `V5678`). We can construct a similar walk by first traversing edges `V1234` $\rightarrow$ `V2346` $\rightarrow$ `V3467`,

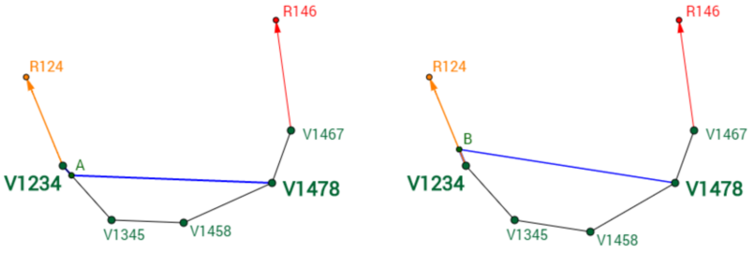

Fig. 4. Getting from `V1478` to `V1234` in at most 2 steps.

and then taking a maximal step in the circuit direction arising from the edge connecting `V1467` and `V1678`. Here we stay in the 2-face determined by facets `6` and `7`. We can then arrive at `V5678` in at most two steps from `V3467`. □

This result illustrates that the first counterexample to the unbounded Hirsch conjecture does have a circuit diameter satisfying the Hirsch bound. Note that this walk used actual edge directions, and so is more restrictive than circuit directions. We further remark that we computed the circuit diameter for $\tilde{U}_4$ using brute force and it was 4 [6]. We do not know if it could be lower in some other representation.

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