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Local resilience of spanning subgraphs in sparse random graphs

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Abstract

For each real $\gamma > 0$ and integers $\Delta \geq 2$ and $k \geq 1$, we prove that there exist constants $\beta > 0$ and $C > 0$ such that for all $p \geq C(\log n/n)^{1/\Delta}$ the random graph $G(n,p)$ asymptotically almost surely contains - even after an adversary deletes an arbitrary $(1/k - \gamma)$-fraction of the edges at every vertex - a copy of every n-vertex graph with maximum degree at most Δ, bandwidth at most βn and at least $C \max\{p^{-2}, p^{-1} \log n\}$ vertices not in triangles.

Keywords: extremal graph theory, random graphs, sparse regularity, resilience

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1 Introduction

In this paper we study graphs that contain every graph from a particular class of graphs in a robust manner. By this we mean that we can still find a copy even after an adversary has deleted a certain proportion of the edges at every vertex. To measure this robustness, we use the following concept of resilience.

Let $\mathcal{P}$ be a monotone increasing graph property and let G be a graph in $\mathcal{P}$. The *local resilience* of a graph G with respect to $\mathcal{P}$ is the minimum $r \in \mathbb{R}$ such that by deleting at each vertex $v \in V(G)$ at most $r \deg(v)$ edges one can obtain a graph not in $\mathcal{P}$. Using this notion, the classic theorem of Dirac [9] implies that the local resilience of K_n with respect to Hamiltonicity is $1/2 - o(1)$. There is a series of other well-known results that can be restated in terms of local resilience of *complete graphs* with respect to containing spanning subgraphs with bounded maximum degree, such as powers of Hamilton cycles, trees, clique-factors, and H-factors (see e.g. [11] for a survey). Schacht and two of the current authors [6] extended these results to families of graphs with sublinear bandwidth, where a graph is defined to have bandwidth at most b if there is a labelling of its vertex set by integers $1, \ldots, n$ such that $|i - j| \leq b$ for every edge $\{i, j\}$.

Theorem 1.1 **([6])** *For each $\gamma > 0$, $\Delta \geq 2$, and $k \geq 1$, there exist constants $\beta > 0$ and $n_0 \geq 1$ such that for every $n \geq n_0$ the following holds. If G is a graph on n vertices with minimum degree $\delta(G) \geq \left(\frac{k-1}{k} + \gamma\right) n$ and if H is a k-colourable graph on n vertices with $\Delta(H) \leq \Delta$, bandwidth at most βn, then G contains a copy of H.*

Many interesting classes of graphs have sublinear bandwidth, for instance the class of all bounded degree planar graphs (see [5]). Thus, Theorem 1.1 applies to quite a large family of graphs and states that the local resilience of the complete graph with respect to containing all bounded degree, k-colourable spanning subgraphs of sublinear bandwidth is $1/k - o(1)$.

Instead of taking the complete graph K_n as the initial graph, one can also study the local resilience of classes of sparser graphs. The Erdős-Rényi random graph model $G(n, p)$ turns out to be quite robust with respect to various properties, where $G(n, p)$ is defined on the vertex set $[n] = \{1, \ldots, n\}$ and each pair of vertices forms an edge randomly and independently of each other with probability p. Huang, Lee, and Sudakov proved in [10] that if p is constant, then the local resilience of $G(n, p)$ with respect to containing all maximum degree bounded bipartite graphs with sublinear bandwidth is asymptotically almost surely (or a.a.s. for short) $1/2 - o(1)$. For much sparser graphs, Lee and Sudakov showed in [12] that the local resilience of $G(n,p)$ with respect to Hamiltonicity is a.a.s. $1/2 - o(1)$ if $p \gg \log n/n$. Another example is the local resilience of $G(n,p)$ with respect to containing cycles of length at least $(1-\alpha)n$ for any $0 < \alpha < 1/2$ which is a.a.s. $1/2 - o(1)$ if $p \gg 1/n$ as shown by Dellamonica, Kohayakawa, Marciniszyn, and Steger in [8]. Balogh, Csaba, and Samotij [2] proved that the local resilience of $G(n,p)$ with respect to containing copies of all trees T on $(1-\eta)n$ vertices and with $\Delta(T) \leq \Delta$ is also a.a.s. $1/2 - o(1)$ if $p \gg 1/n$.

Recently, Kohayakawa and two of the current authors proved in [4] that a.a.s. the local resilience of $G(n,p)$ with respect to containing all *nearly spanning* bipartite graphs with maximum degree at most Δ and sublinear bandwidth is $1/2-o(1)$ if $p \gg (\log n/n)^{1/\Delta}$. Moreover, Balogh, Lee, and Samotij [3] proved that if $p \gg (\log n/n)^{1/2}$, then a.a.s. $G(n,p)$ has local resilience $1/3 - o(1)$ with respect to a triangle packing that covers all but at most $\mathcal{O}(p^{-2})$ vertices. Furthermore, it is known that one cannot hope for a spanning triangle-factor because Huang, Lee, and Sudakov showed in [10] that for each $\varepsilon > 0$ there exists some constant $p_\varepsilon > 0$ such that for all $0 < p \leq p_\varepsilon$, the random graph $\Gamma = G(n,p)$ contains a.a.s. a spanning subgraph G with $\delta(G) > (1-\varepsilon)np$ such that at least $\varepsilon p^{-2}/3$ vertices of G are not contained in any triangles.

Here we establish a random graph analogue of Theorem 1.1, determining the local resilience of $G(n,p)$ with respect to containing the graphs H from Theorem 1.1 provided that enough vertices of H are not contained in triangles.

Theorem 1.2 *For each $\gamma > 0$, $\Delta \geq 2$, and $k \geq 1$, there exist constants $\beta > 0$ and $C > 0$ such that the following holds asymptotically almost surely for $\Gamma = G(n,p)$ if $p \geq C(\log n/n)^{1/\Delta}$. Let G be a spanning subgraph of Γ with $\delta(G) \geq ((k-1)/k + \gamma)\,pn$ and let H be a k-colourable graph on n vertices with $\Delta(H) \leq \Delta$, bandwidth at most βn and with at least $C\max\{p^{-2}, p^{-1}\log n\}$ vertices not contained in any triangles of H. Then G contains a copy of H.*

In particular, Theorem 1.2 yields the local resilience of $G(n,p)$ with respect to containing spanning grids or cycle-factors. Moreover, the theorem remains true if we allow H to have a few vertices that are coloured with an additional $(k+1)$-st colour. In particular, it can thus be applied to, say, Hamilton cycles on an odd number of vertices.

2 Outline of the proof

The proof of Theorem 1.2 (in the case $\Delta \geq 3$) can be split into five lemmas, four of which we state explicitly in this section. The fifth is the so-called sparse blow up lemma developed by Hàn, Kohayakawa, Person, and two of the current authors in [1]. The lemma is too long and complicated to be stated here in detail, but it serves as a powerful tool for embedding maximum degree bounded spanning graphs into sparse graphs. In particular, given a subgraph $G \subseteq \Gamma = G(n,p)$, where $p \gg (\log n/n)^{1/\Delta}$, with a vertex partition $\mathcal{V}$ and a graph H with maximum degree at most Δ on the same number of vertices as G and with a vertex partition $\mathcal{W}$, the sparse blow up lemma guarantees under certain conditions a spanning embedding of H in G which respects the given partitions. Lemmas 2.2–2.5 deal with by the preparation of vertex partitions of the graphs G and H such that we can apply the sparse blow up lemma to those subgraphs of G and H that were not dealt with some manual pre-embedding process. We remark that the proofs of Lemmas 2.2, 2.3 and 2.5 borrow ideas from the techniques developed in [4,6], but are technically more involved because of the requirements posed by the sparse blow-up lemma.

Before stating these lemmas, we introduce some necessary definitions. Our proofs rely heavily on the concept of regular pairs.

Definition 2.1 A pair (X,Y) is called $(\varepsilon,d,p)_G$*-regular* if for every $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| \geq \varepsilon|X|$ and $|Y'| \geq \varepsilon|Y|$ we have

$$e_G(X',Y') \geq (d-\varepsilon)p|X'||Y'|.$$

If additionally we have $|N_G(x,Y)| \geq dp|Y|$ and $|N_G(y,X)| \geq dp|X|$ for every $x \in X$ and $y \in Y$, then the pair (X,Y) is called $(\varepsilon,d,p)_G$*-super-regular.*

Let $r,k \geq 1$ and let B_r^k be the graph on kr vertices obtained from a path on r vertices by replacing every vertex by a clique of size k and by replacing every edge by a complete bipartite graph minus a perfect matching. More precisely, we define B_r^k as

$$V(B_r^k) := [r] \times [k]$$

and for every $j \neq j' \in [k]$

$$\{(i,j),(i',j')\} \in E(B_r^k) \quad \text{if and only if} \quad i = i' \text{ or } |i-i'| = 1.$$

Let $K_r^k \subseteq B_r^k$ be the spanning subgraph of B_r^k that is the disjoint union of r complete graphs on k vertices given by the components $B_r^k[\{(i,1),\ldots,(i,k)\}]$ for each $i \in [r]$. A vertex partition $\mathcal{V}' = \{V_{i,j}\}_{i\in[r],j\in[k]}$ is called k*-equitable* if $\big||V_{i,j}| - |V_{i,j'}|\big| \leq 1$ for every $i \in [r]$ and $j,j' \in [k]$. Similarly, an integer partition $\{n_{i,j}\}_{i\in[r],j\in[k]}$ of n (meaning that $n_{i,j} \in \mathbb{N}_0$ for every $i \in [r], j \in [k]$

and $\sum_{i\in[r]j\in[k]} n_{i,j} = n$) is *k-equitable* if $|n_{i,j} - n_{i,j'}| \leq 1$ for every $i \in [r]$ and $j, j' \in [k]$.

Now we are in the position to state our first lemma, which suggests a partition of G that satisfies some specific regularity properties and passes this structure to Lemma 2.3, which which will try to find a partition of H that is similar to this one.

Lemma 2.2 (Lemma for G) *For each real $\gamma > 0$ and integers $k \geq 2$ and $r_0 \geq 1$ there exists $d > 0$ such that for every $\varepsilon \in \left(0, \frac{1}{2k}\right)$ there exist $r_1 \geq 1$, $\tilde{C} > 0$, and $C' > 0$ such that the following holds a.a.s. for $\Gamma = G(n,p)$ if $p \geq C' (\log n/n)^{1/2}$. Let $G = (V, E)$ be a spanning subgraph of Γ with $\delta(G) \geq \left(\frac{k-1}{k} + \gamma\right) pn$. Then there exist $r \geq 1$ such that $r_0 \leq kr \leq r_1$, a subset $V_0 \subseteq V$ with $|V_0| \leq \tilde{C} \max\{p^{-2}, p^{-1} \log n\}$, a k-equitable vertex partition $\mathcal{V} = \{V_{i,j}\}_{i\in[r],j\in[k]}$ of $V(G) \setminus V_0$, and a graph R^k_r on the vertex set $[r] \times [k]$ with $K^k_r \subseteq B^k_r \subseteq R^k_r$ such that $\delta(R^k_r) \geq \left(\frac{k-1}{k} + \frac{\gamma}{2}\right) kr$, and such that the following is true.*

(G1) $\frac{n}{4kr} \leq |V_{i,j}| \leq \frac{4n}{kr}$ *for every $i \in [r]$ and $j \in [k]$,*

(G2) *$\mathcal{V}$ is $(\varepsilon, d, p)_G$-regular on R^k_r and $(\varepsilon, d, p)_G$-super-regular on K^k_r,*

(G3) *$\left(N_\Gamma(v, V_{i,j}), V_{i',j'}\right)$ and $\left(N_\Gamma(v', V_{i,j}), N_\Gamma(v', V_{i',j'})\right)$ are $(\varepsilon, d, p)_G$-regular pairs for every edge $\{(i,j),(i',j')\} \in E(R^k_r)$, $v \in V \setminus (V_0 \cup V_{i,j})$, and $v' \in V \setminus (V_0 \cup V_{i,j} \cup V_{i',j'})$, and*

(G4) *we have $(1-\varepsilon)p|V_{i,j}| \leq |N_\Gamma(v, V_{i,j})| \leq (1+\varepsilon)p|V_{i,j}|$ for every $i \in [r]$, $j \in [k]$ and every vertex $v \in V \setminus V_0$.*

The idea of the proof of Lemma 2.2 can be summarized as follows. First, we apply a minimum degree version of the sparse regularity lemma (see e.g. [4]) to G, which yields a regular equipartition, the reduced graph R of which has minimum degree greater than $((k-1)/k + 2\gamma/3)\,|V(R)|$. Hence, by Theorem 1.1 we know that $K^k_r \subseteq B^k_r \subseteq R$, where r is uniquely determined by $rk + b = |V(R)|$ with some $0 \leq b < k$. Then, by a careful redistribution of vertices of G and by exploiting properties of regular pairs, we can guarantee a vertex partition as stated in the lemma.

After Lemma 2.2 has constructed a regular partition $\mathcal{V}$ of $V(G)$, the second lemma deals with the graph H that we would like to find as a subgraph of G. More precisely, Lemma 2.3 provides a homomorphism f from the graph H to the reduced graph R^k_r given by Lemma 2.2 which has among others the following properties. The edges of H are mapped to the edges of R^k_r where most of the edges of H are assigned to edges of the clique factor $K^k_r \subseteq R^k_r$. Furthermore, the number of vertices of H mapped to a vertex of R^k_r only differs by a small factor from the size of the corresponding cluster of $\mathcal{V}$. The lemma further guarantees that the first $\sqrt{\beta}n$ vertices of the bandwidth ordering of $V(H)$ are mapped to one component of K^k_r.

Lemma 2.3 (Lemma for H) *Given $r \geq 1$, $k \geq 1$, and $\xi > 0$, let β satisfy $\beta \leq \xi^2/(1200k)$. Let H be a k-colourable graph on n vertices that has a labelling $\mathcal{L}$ of its vertex set of bandwidth at most βn and let F denote the set of the first $\sqrt{\beta}n$ vertices with respect to $\mathcal{L}$. Furthermore, let R^k_r be a graph on vertex set $[r]\times[k]$ with $K^k_r \subseteq B^k_r \subseteq R^k_r$ such that $\delta(R^k_r) > (k-1)r$. Then, given a k-equitable integer partition $\{m_{i,j}\}_{i\in[r],j\in[k]}$ of n with $m_{i,j} \geq 12\beta n$ for every $(i,j) \in [r]\times[k]$, there exists a mapping $f\colon V(H) \to [r]\times[k]$ and a set of special vertices $X \subseteq V(H)$ with the following properties, where $W_{i,j} := f^{-1}(i,j)$.*

(H1) $|X \cap W_{i,j}| \leq \xi n$, $|N_H(X \cap W_{i,j}) \cap W_{i',j'}| \leq \xi n$ *for every* $i, i' \in [r]$ *and* $j, j' \in [k]$,

(H2) $m_{i,j} - \xi n \leq |W_{i,j}| \leq m_{i,j} + \xi n$ *for every* $i \in [r]$ *and* $j \in [k]$,

(H3) for every edge $\{x, y\} \in E(H)$ *we have* $\{f(x), f(y)\} \in E(R^k_r)$,

(H4) for every $\{x, y\} \in E(H) \setminus E(H[X])$ *we have* $\{f(x), f(y)\} \in E(K^k_r)$,

(H5) $F \subseteq \bigcup_{j\in[k]} W_{1,j}$.

Lemma 2.3 without Property (H5) is a special case of Lemma 8 in [7] and Property (H5) can be derived from its proof.

After having assigned all vertices of H to the clusters of $\mathcal{V}$ using Lemma 2.3, we aim to apply the sparse blow up lemma in order to embed H onto G. However, we need to tackle two problems first that will be resolved by Lemma 2.4 and Lemma 2.5. First of all, the vertices of the exceptional set V_0 were disregarded in the assignment. Therefore, we need to manually pre-embed vertices of H onto all vertices in V_0. For this, we use vertices in H that are not in triangles, that are pairwise far apart from each other and that are contained in the first βn vertices of the bandwidth ordering $\mathcal{L}$ of $V(H)$. We also directly pre-embed all neighbours of these H-vertices. In this way, we create image restrictions for the embedding of their neighbours.

The next lemma ensures by choosing $W \subseteq N_G(v)$ that we find for any vertex $v \in V_0$ at least Δ many G-neighbours such that if we embed vertices of H onto these vertices, the resulting image restrictions satisfy all necessary conditions for the sparse blow up lemma.

Lemma 2.4 (Common neighbourhood lemma) *For each $d > 0$, $k \geq 1$, and $\Delta \geq 2$ there exists $\alpha > 0$ such that for every $\varepsilon^* \in (0,1)$ there exists $\varepsilon_0 > 0$ such that for every $r \geq 1$ there exists $C^* > 0$ such that if $p \geq C^* (\log n/n)^{1/\Delta}$,* *then $\Gamma = G(n,p)$ a.a.s. satisfies the following for any $0 < \varepsilon < \varepsilon_0$.*

Let $G = (V,E)$ be a (not necessarily spanning) subgraph of Γ and $\{V_i\}_{i\in[k]} \cup W$ a vertex partition of a subset of V such that the following is true for every $i, i' \in [k]$.

(V1) $|V_i| \geq \frac{n}{4r}$,

(V2) (V_i, V_j) *is* $(\varepsilon, d, p)_G$*-regular,*

(V3) $|W| \geq \frac{pn}{2r}$, *and*

(V4) $|N_G(w, V_i)| \geq dp|V_i|$ *for every* $w \in W$.

Then there exists a tuple $(w_1, \ldots, w_\Delta) \in \binom{W}{\Delta}$ such that for every $\Lambda, \Lambda^* \subseteq [\Delta]$, $\Lambda, \Lambda^* \neq \varnothing$, and every $i \neq i' \in [k]$ we have

(W1) $\left(\bigcap_{j\in\Lambda} N_\Gamma(w_j, V_i), V_{i'}\right)$ *is* $(\varepsilon^*, d, p)_G$*-regular if* $|\Lambda| < \Delta$,

(W2) $|\bigcap_{j\in\Lambda} N_G(w_j, V_i)| \geq \alpha p^{|\Lambda|}|V_i|$,

(W3) $(1-\varepsilon^*)p^{|\Lambda|}|V_i| \leq |\bigcap_{j\in\Lambda} N_\Gamma(w_j, V_i)| \leq (1+\varepsilon^*)p^{|\Lambda|}|V_i|$, *and*

(W4) $\left(\bigcap_{j\in\Lambda} N_\Gamma(w_j, V_i), \bigcap_{j^*\in\Lambda^*} N_\Gamma(w_{j^*}, V_{i'})\right)$ *is* $(\varepsilon^*, d, p)_G$*-regular if* $|\Lambda|, |\Lambda^*| < \Delta$ *and either* $\Lambda \cap \Lambda^* = \varnothing$ *or* $\Delta \geq 3$ *or both.*

Lemma 2.4 is proved by induction and the proof mainly uses Chernoff's inequality and the so-called one-sided and two-sided regularity inheritance lemmas (see [1]).

Let H' and G' denote the subgraphs of H and G that result from removing all vertices that were used in the pre-embedding process. As a last step before finally applying the sparse blow up lemma, the clusters in $\mathcal{V}\big|_{G'}$ need to be adjusted to the sizes of $W_{i,j}\big|_{H'}$. The next and last lemma assures that this is indeed possible and that after this redistribution important regularity properties for the application of the sparse blow up lemma still hold.

Lemma 2.5 (Balancing lemma) *For all integers $k \geq 1$, $r_1, \Delta \geq 1$, and reals $\gamma, d > 0$ and $0 < \varepsilon < \min\{d, 1/(2k)\}$ there exist $\xi > 0$ and $\hat{C} > 0$ such that the following is true for every $p \geq \hat{C}\,(\log n/n)^{1/2}$ and every $10\gamma^{-1} \leq r \leq r_1$ provided that n is large enough. Let Γ be a graph on the vertex set $[n]$ and let $G = (V,E) \subseteq \Gamma$ be a (not necessarily spanning) subgraph with vertex partition $\mathcal{V} = \{V_{i,j}\}_{i\in[r],j\in[k]}$ that satisfies $n/(8kr) \leq |V_{i,j}| \leq 4n/(kr)$ for each $i \in [r]$, $j \in [k]$. Let $\{n_{i,j}\}_{i\in[r],j\in[k]}$ be an integer partition of $\sum_{i\in[r],j\in[k]} |V_{i,j}|$. Let R_r^k be a graph on the vertex set $[r]\times[k]$ with minimum degree $\delta(R_r^k) \geq \big((k-1)/k + \gamma/2\big)kr$ such that $K_r^k \subseteq B_r^k \subseteq R_r^k$. Suppose that the partition $\mathcal{V}$ satisfies the following properties for each $i \in [r]$, each $j \neq j' \in [k]$, and each $v \in V$.*

(B1) We have $n_{i,j} - \xi n \le |V_{i,j}| \le n_{i,j} + \xi n$*,*

(B2) $\mathcal{V}$ *is* $\left(\frac{\varepsilon}{4}, d, p\right)_G$*-regular on* R_r^k *and* $\left(\frac{\varepsilon}{4}, d, p\right)_G$*-super-regular on* K_r^k*,*

(B3) both $\big(N_\Gamma(v, V_{i,j}), V_{i,j'}\big)$ *and* $\big(N_\Gamma(v, V_{i,j}), N_\Gamma(v, V_{i,j'})\big)$ *are* $\left(\frac{\varepsilon}{4}, d, p\right)_G$*-regular pairs, and*

(B4) we have $|N_\Gamma(v, V_{i,j})| = \big(1 \pm \frac{\varepsilon}{4}\big)p|V_{i,j}|$*.*

Then, there exists a partition $\mathcal{V}' = \{V'_{i,j}\}_{i\in[r],j\in[k]}$ *of* V *such that the following properties hold for each* $i \in [r]$*, each* $j \ne j' \in [k]$*, and each* $v \in V$*.*

(B1') We have $|V'_{i,j}| = n_{i,j}$*,*

(B2') $\mathcal{V}'$ *is* $(\varepsilon, d, p)_G$*-regular on* R_r^k *and* $(\varepsilon, d, p)_G$*-super-regular on* K_r^k*,*

(B3') both $\big(N_\Gamma(v, V'_{i,j}), V'_{i,j'}\big)$ *and* $\big(N_\Gamma(v, V'_{i,j}), N_\Gamma(v, V'_{i,j'})\big)$ *are* $(\varepsilon, d, p)_G$*-regular pairs, and*

(B4') for each $1 \le s \le \Delta$ *and vertices* $v_1, \ldots, v_s \in [n]$ *we have*

$$\Big|\bigcap_{\ell\in[s]} N_\Gamma(v_\ell, V_{i,j}) \triangle \bigcap_{\ell\in[s]} N_\Gamma(v_\ell, V'_{i,j})\Big| \le \tfrac{\varepsilon}{100kr}\deg_\Gamma(v_1, \ldots, v_s) + \hat{C}\log n\,.$$

As a last step we apply the sparse blow up lemma to the vertex partition of G' given by Lemma 2.5 and to the vertex partition of H given by Lemma 2.3 restricted to H' while respecting the image restrictions that resulted from the pre-embedding process.

References

[1] Allen, P., J. Böttcher, H. Hàn, Y. Kohayakawa and Y. Person, *Blow-up lemmas for sparse graphs*, In preparation.

[2] Balogh, J., B. Csaba and W. Samotij, *Local resilience of almost spanning trees in random graphs*, Random Structures & Algorithms **38** (2011), 121–139.

[3] Balogh, J., C. Lee and W. Samotij, *Corrádi and Hajnal's theorem for sparse random graphs*, Combinatorics, Probability and Computing **21** (2012), 23–55.

[4] Böttcher, J., Y. Kohayakawa and A. Taraz, *Almost spanning subgraphs of random graphs after adversarial edge removal*, Combinatorics, Probability and Computing **22** (2013), 639–683.

[5] Böttcher, J., K. Pruessmann, A. Taraz and A. Würfl, *Bandwidth, expansion, treewidth, separators and universality for bounded-degree graphs*, European Journal of Combinatorics **31** (2010), 1217–1227.

[6] Böttcher, J., M. Schacht and A. Taraz, *Proof of the bandwidth conjecture of Bollobás and Komlós*, Mathematische Annalen **343** (2009), 175–205.

[7] Böttcher, J., A. Taraz and A. Würfl, *Spanning embeddings of arrangeable graphs with sublinear bandwidth* Submitted, arXiv:1305.2078, 2013.

[8] Dellamonica, D., Y. Kohayakawa, M. Marciniszyn and A. Steger, *On the resilience of long cycles in random graphs*, Electronic Journal of Combinatorics **15** (2008), R32.

[9] Dirac, G. A., *Some theorems on abstract graphs*, Proceedings of the London Mathematical Society **3** (1952), 69–81.

[10] Huang, H., C. Lee and B. Sudakov, *Bandwidth theorem for random graphs*, Journal of Combinatorial Theory, Series B **102** (2012), 14–37.

[11] Kühn, D., and D. Osthus, *Embedding large subgraphs into dense graphs* in Surveys in Combinatorics (editors S. Huczynka, J. Mitchel, C. Roney-Dougal), London Math. Soc. Lecture Notes **365** (2009), 137–167.

[12] Lee, C., and B. Sudakov, *Dirac's theorem for random graphs*, Random Structures & Algorithms **41** (2012), 293–305.