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On the generalised colouring numbers of graphs that exclude a fixed minor

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Abstract

The colouring number $\mathrm{col}(G)$ of a graph G is the minimum integer k such that there exists a linear ordering of the vertices of G in which each vertex v has back-degree at most k, i.e. v has at most k neighbours u with $u < v$. The colouring number is a structural measure that measures the edge density of subgraphs of G. For $r \geq 1$, the numbers $\mathrm{col}_r(G)$ and $\mathrm{wcol}_r(G)$ generalise the colouring number, where $\mathrm{col}_1(G)$ and $\mathrm{wcol}_1(G)$ are equivalent to $\mathrm{col}(G)$. For increasing values of r these measures converge to the well-known structural measures tree-width and tree-depth. For an n-vertex graph, $\mathrm{col}_n(G)$ is equal to the tree-width of G and $\mathrm{wcol}_n(G)$ is equal to the tree-depth of G.

We show that if G excludes K_t as a minor, then $\mathrm{col}_r(G) \leq \binom{t}{2} \cdot (2r+1)$ and $\mathrm{wcol}_r(G) \leq \binom{t}{2}^r \cdot (2r+1)$.

It is easily observed that if G is planar, then $\mathrm{col}_r(G) \leq 5r+3$. The technically most demanding part of the paper is to show that for those graphs, $\mathrm{wcol}_r(G) \leq 5r^5$. These results generalise to bounded genus graphs, i.e. if G is of genus g, then $\mathrm{col}_r(G) \leq (2g+3)(2r+1)$ and $\mathrm{wcol}_r(G) \leq 2g(2r+1) + 5r^5$.

Keywords: Generalised colouring numbers, planar graphs, excluded minors

1 Preliminaries

Generalised colouring numbers have been introduced by Kierstead and Yang in the context of colouring games and marking games on graphs [7], and received much attention recently, as they can be used to characterise nowhere dense classes of graphs [9,11]. They find algorithmic applications e.g. for the constant factor approximation of r-dominating sets on bounded expansion classes [4] or for the construction of sparse neighbourhood covers on nowhere dense classes [6]. Let us quickly provide the required background.

All graphs in this paper are simple and undirected. For a graph G, we write $\Pi(G)$ for the set of linear orders on $V(G)$. A vertex u is *weakly r-reachable* from v with respect to an order $\leq \in \Pi(G)$, if there exists a path P of length ℓ, $0 \leq \ell \leq r$, between u and v such that u is minimum in $V(P)$ with respect

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to $\leq$. Let $\mathrm{WReach}_r[G, \leq, v]$ be the set of vertices that are weakly r-reachable from v with respect to $\leq$.

A vertex u is *strongly r-reachable* from v with respect to an order $\leq \in \Pi(G)$, if there is a path P of length ℓ, $0 \leq \ell \leq r$, connecting u and v such that $u \leq v$ and such that all inner vertices w of P satisfy $w > v$. Let $\mathrm{SReach}_r[G, \leq, v]$ be the set of vertices that are strongly r-reachable from v with respect to $\leq$.

The *weak r-colouring number* $\mathrm{wcol}_r(G)$ of G is defined as

$$\mathrm{wcol}_r(G) = \min_{\leq \in \Pi(G)} \max_{v \in V(G)} |\mathrm{WReach}_r[G, \leq, v]|,$$

and the *r-colouring number* $\mathrm{col}_r(G)$ of G is defined as

$$\mathrm{col}_r(G) = \min_{\leq \in \Pi(G)} \max_{v \in V(G)} |\mathrm{SReach}_r[G, \leq, v]|.$$

As noticed in [7], these invariants are related by the inequalities $\mathrm{col}_r(G) \leq \mathrm{wcol}_r(G) \leq (\mathrm{col}_r(G))^r$. Using probabilistic arguments, Zhu [11] was the first to give a non-trivial bound for $\mathrm{col}_r(G)$ in terms of the densities of shallow minors of G, and his results were improved in [5]. In particular, when a graph G excludes a complete graph K_t as a minor, one deduces an upper bound for $\mathrm{col}_r(G)$, which grows as fast as $(c \cdot r \cdot t)^r$ for some constant c. One of our main results is a dramatic decrease of this bound: we prove that if G excludes K_t as a minor, then $\mathrm{col}_r(G) \leq \binom{t}{2} \cdot (2r+1)$ and $\mathrm{wcol}_r(G) \leq \binom{t}{2}^r \cdot (2r+1)$. The second result is that for graphs G with genus g, $\mathrm{wcol}_r(G) \leq 2g(2r+1) + 5r^5$.

2 The r-colouring number of classes that exclude a minor

Let G be a graph. We call a path P in G a *shortest path* if there is no shorter path between its endpoints. A *shortest paths decomposition* (compare to [1]) of G is a sequence $P_0, \ldots, P_\ell$ of paths such that $\bigcup_{i=0}^{\ell} V(P_i) = V(G)$, defined inductively as follows. Let P_0 be an arbitrary shortest path in G and let $G_0 := P_0$. For $i > 0$, let $P_i = v_0, \ldots, v_n$ be a shortest path in $G - E(G_{i-1})$ such that $V(G_{i-1}) \cap V(P_i) \subseteq \{v_0, v_n\}$ and let $G_i := G_{i-1} + P_i$ (the graph induced by $V(G_{i-1}) \cup P_i$). Let $\mathcal{C}_i$ be the set of components of $G - G_i$. The separating number of a component $C \in \mathcal{C}_i$ is the minimum number s of paths $Q_1, \ldots, Q_s \in \{P_0, \ldots, P_\ell\}$ such that $\bigcup_{1 \leq j \leq s} V(Q_s)$ separates C from $G - G_i$.

The *width* of $P_0, \ldots, P_\ell$ is the maximum separating number over all i and all $C \in \mathcal{C}_i$.

Theorem 2.1

(1) If G has genus g, then G has a shortest paths decomposition of width $2g+2$ [2,10].

(2) If G excludes K_t as a minor, then G has a shortest paths decomposition of width $\left(\binom{t}{2}-1\right)$ [3].

From a shortest paths decomposition $P_0, \ldots, P_\ell$, we define a linear order $\sqsubseteq$ on $V(G)$ as follows. For $v, w \in V(G)$, set $v \sqsubseteq w$ if $v \in V(P_i) = v_0, \ldots, v_n$, $w \in V(P_j) \setminus V(P_i)$ and $i < j$, or $i = j$, $v = v_x$, $w = v_y$ and $x < y$. We write $P(v)$ for the path P_m with minimum index m such that $v \in V(P_m)$. In the following, let $v \in V(G)$ and let m be such that $P(v) = P_m$. The proof of Theorem 2.5 is based on the following observations.

Lemma 2.2 *Let P be a shortest path in a graph G. Then $|N_r(v) \cap V(P)| \leq 2r+1$ for all $v \in V(G)$, where $N_r(v)$ denotes the r-neighbourhood of v (containing v).*

Lemma 2.3 $\mathrm{WReach}_r[G, \sqsubseteq, v] \subseteq V(G_m)$.

Lemma 2.4 *Let C be a component of $G - G_i$ for some $i < m$ which does not contain v. Then $v \notin \mathrm{WReach}_r[G, \sqsubseteq, u]$ for all $u \in V(C)$.*

Theorem 2.5 *If G has a shortest paths decomposition of width k, then*

(1) $\mathrm{col}_r(G) \leq (k+1) \cdot (2r+1)$, *and*

(2) $\mathrm{wcol}_r(G) \leq (k+1)^r \cdot (2r+1)$.

Proof. Consider the component C in $G - G_{m-1}$ which contains v. It is separated by k paths whose vertices are the only strongly reachable vertices from v. Furthermore, at most $r+1$ vertices on $P(v)$ are reachable. For wcol, the argument is similar; the number of paths can be bounded by a simple induction on r. □

Corollary 2.6 *If G excludes K_t as a minor, then* $\mathrm{col}_r(G) \leq \binom{t}{2} \cdot (2r+1)$ *and* $\mathrm{wcol}_r(G) \leq \binom{t}{2}^r \cdot (2r+1)$.

3 The weak r-colouring number of planar graphs

We fix a planar graph G and as adding edges to a graph can only increase its weak r-colouring number, we may assume without loss of generality that G is maximally planar and hence 3-connected. It holds that $\mathrm{wcol}_1(G)$ is equal to the degeneracy of G plus one, so we always assume that $r \geq 2$.

We inductively define a shortest paths decomposition of G. Along with the construction we guarantee that for all i, if C is a component of $G - G_i$, then there are at most two paths P_j and P_ℓ with $j \leq \ell \leq i$ such that C is separated from $V(G_i)$ in G by $V(P_j) \cup V(P_\ell)$. We write $S_1(C) = P_j$ and $S_2(C) = P_\ell$ for the least possible j and ℓ with that property and call S_1, S_2 the *separating paths* of the component C. Note that if S_1 alone separates C, then $S_1 = S_2$. As G is 3-connected, C has at least three neighbours in $V(S_1) \cup V(S_2)$. Hence some $P \in \{S_1, S_2\}$ has at least two *C-neighbours*, i.e. vertices which are adjacent to a vertex of C.

Our construction. The path P_0 is an arbitrary shortest path in G. Let $i > 0$ and assume $P_0, \dots, P_{i-1}$ have been defined such that for each component C of $G - V(G_{i-1})$ there are at most two separating paths $S_1(C)$ and $S_2(C)$. Let C be a component of $G - V(G_{i-1})$. Then some $P = w_0, \dots, w_\ell \in \{S_1, S_2\}$ has two C-neighbours. Let $w_{\min}$ ($w_{\max}$) be the C-neighbours of P with the least (greatest) index. We define P_i as a shortest path between $w_{\min}$ and $w_{\max}$ in $G - E(G_{i-1})$ with internal vertices from C (note that P_i has an internal vertex as P is a shortest path in $G - G_{i-1}$). We say that P_i is *anchored* at P. The procedure stops when no $v \in V(G) \setminus V(G_{i-1})$ can be found, hence when $V(G_i) = V(G)$, i.e. when a shortest paths decomposition of G was found.

The next lemma follows easily by the Jordan Curve Theorem and our choice of anchoring new paths at minimal and maximal C-neighbours.

Lemma 3.1 *For $i > 0$, if C is a component of $G - G_i$, then there are two paths P_j and P_ℓ with $j \leq \ell \leq i$ such that C is separated from $V(G_i)$ in G by $V(P_j) \cup V(P_\ell)$.*

Lemma 3.2 *Let C be a component of $G - G_i$. Then $P \in \{S_1(C), S_2(C)\}$ (for $P \neq P_0$) has an inner vertex which is a C-neighbour.*

Proof. S_1 and S_2 are paths with minimal indexes with the separator property. Their endpoints lie on paths with smaller indices. □

Let P be a path from the shortest paths decomposition. The *chain* $\chi(P)$ of P is the sequence $Q_0, \ldots, Q_n$ of paths from the shortest paths decomposition where $Q_0 = P$, $Q_n = P_0$ and for $0 < j < n$, $Q_j = P'$ if and only Q_{j-1} is anchored at P'. For $w \in V(G)$, $\chi(w)$ is defined as $\chi(P(w))$. Note that any two chains $\chi_1 = U_1, \ldots, U_m$ and $\chi_2 = U'_1, \ldots, U'_n$ coincide from some path on. The *meeting path* of χ_1 and χ_2 is the path P_i such that $P_i = U_i = U'_j$ for the least i (and j).

Lemma 3.3 *In the subgraph induced by the vertices of* $\chi(v)$*, there are at most* r^3 *weakly* r*-reachable paths from* v*.*

Proof. Let $0 \leq i \leq r$ and let $P_{j(i)}$ be the path of the chain with the minimum index such that $P_{j(i)}$ is weakly reachable from v in i steps. Let χ_i be the chain that contains only the paths with index at least as large as $j(i)$ (in the chain order). We show by induction on i that there are at most $i \cdot r$ pairs of endpoints of paths from χ_i which are weakly r-reachable from v. Clearly, we reach only $P(v)$ in 0 steps. Let $i > 0$ and assume that the claim holds for all $\ell < i$. We can reach only an inner vertex on $P_{j(i-1)}$ in $i - 1$ steps (if we could reach an endpoint, then $j(i-1)$ would not be the minimal index).

We count the tuples of endpoints of paths which lie in χ_i and which are weakly reachable in $r - i$ steps from some inner vertex v' of $P_{j(i-1)}$. As $P_{j(i-1)}$ separates $\chi_{i-1} - P_{j(i-1)}$ from $P_{J(i)}$, the path $P_{j(i)}$ is reached in one step from $P_{j(i-1)}$ and gives us one additional endpoint tuple (or $P_{j(i-1)} = P_{j(i)}$ and we are done in this step).

Now one endpoint, say x, of $P_{j(i-1)}$ is an endpoint of $P_{j(i)}$. Otherwise let P be the path at which $P_{j(i-1)}$ is anchored. Then P separates $P_{j(i-1)}$ from $P_{j(i)}$ and $P_{j(i)}$ is not reachable from $P_{j(i-1)}$ in one step.

All paths P from χ that are weakly reachable from $P_{j(i-1)}$ in χ_i have x as an endpoint, otherwise P separates $P_{j(i-1)}$ from $P_{j(i)}$. Thus we reach at most $r - i$ additional paths with a different second endpoint in $r - i$ steps from $P_{j(i-1)}$.

To conclude the proof, note that χ_r contains all weakly r-reachable paths. For every pair (x, y) of endpoints, there are at most r weakly reachable paths with those endpoints (x, y). This is because every such path P separates the chain and x and y are smaller than every inner vertex of P with respect to $\sqsubseteq$.

Hence in χ_i there are at most $i \cdot r^2$ weakly r-reachable paths. □

Lemma 3.4 *There are at most* $2r^4$ *weakly reachable paths.*

Proof. For a chain $\chi = Q_1, \ldots, Q_m$, let $\sim\chi$ be the chain $Q_2, \ldots, Q_m$. For $i > 0$ and a path P_i from the decomposition let $C(P_i)$ be the component of $G - G_{i-1}$ which contains an inner vertex of P_i (this is well defined). For $j = 1, 2$, let $\chi_j(P_i) = \chi(S_j(C))$, i.e. the chains of the separating paths.

Then for every $S \in \{S_1(C), S_2(C)\}$, $\chi_1(S) \in \{\sim\chi_1(P_i), \sim\chi_2(P_i)\}$ or $\chi_2(S) \in \{\sim\chi_1(P_i), \sim\chi_2(P_i)\}$. As one needs at least one step to change a chain, we can reach at most $2r$ chains in r steps. The result follows by Lemma 3.3. □

Theorem 3.5 *If G is planar, then* $\mathrm{wcol}_r(G) \leq 2r^4 \cdot (2r+1) \in \mathcal{O}(r^5)$.

It is well known (see e.g. [8], Lemma 4.2.4, or [10]) that for a graph of genus $g > 0$, there exists a non-separating cycle C which consists of two shortest paths such that $G - C$ has genus $g - 1$. We can eliminate those cycles in the first place and obtain the planar case.

Theorem 3.6 *If G is of genus g, then* $\mathrm{wcol}_r(G) \leq (2g + 2r^4)(2r+1)$.

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