Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in DISCRETE MATHEMATICS

www.elsevier.com/locate/endm

Structural Limits and Approximations of Mappings

Jaroslav Nešetřil [1,2]

Computer Science Institute of Charles University (IUUK)
Prague, Czech Republic

Patrice Ossona de Mendez [1,3]

Centre d'Analyse et de Mathématiques Sociales (CNRS, UMR 8557)
Paris, France
and
Computer Science Institute of Charles University (IUUK)
Prague, Czech Republic

Abstract

We extend the general framework of structural limits from graphs and relational structures to finite structures (including function symbols). For perhaps the simplest model of this type — sets with single unary function — we determine limit objects with respect to the three main fragments of first order. In each of these cases we solve an analog of Aldous-Lyons conjecture. This builds on the experience gained when studying limits of sequences of trees.

[1] Supported by grant ERCCZ LL-1201 and CE-ITI P202/12/G061, and by the European Associated Laboratory "Structures in Combinatorics" (LEA STRUCO)

[2] Email: nesetril@iuuk.mff.cuni.cz

[3] Email: pom@ehess.fr

1 Introduction and Previous Work

Limit objects of convergent sequences of combinatorial objects recently gained much interest [9]. This was initiated by the theory of limits of dense graphs [10] and of limits of graphs with bounded degrees [3], followed by the study of limits of hypergraphs [5], of permutations [7], etc. The unifying framework of *structural convergence* has been proposed by the authors for general combinatorial structures [11], based on a blend of combinatorics, probability theory, model theory, and functional analysis, which may be outlined as follows:

Recall that a σ-structure $\mathbf{A}$ is defined by its *domain* A, its *signature* σ (which is a set of symbols of relations and functions with their arities, and the interpretation in A of all the relations and functions in σ. A *relational structure* is a structure whose signature only contains relational symbols, while an *algebra* is a structure whose signature only contains function symbols. Let X be a fragment of first-order logic (in the language defined from the signature σ). A sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ of σ-structures is X-*convergent* if, for every formula $\phi \in X$, the probability $\langle \phi, \mathbf{A}_n \rangle$ that ϕ is satisfied for a random assignment of elements of A_n to the free variables converges as n grows to infinity. Three fragments of first-order logic are of specific interest. They are all considered in this paper:

- the fragment QF of quantifier-free formulas, which naturally extends the notion of *left convergence* of dense graphs of Lovász and Szegedy [10];
- the fragment $\mathrm{FO}^{\mathrm{local}}$ of local formulas (that is of formulas whose satisfaction only depends on a r-neighborhood of the free variables), which naturally extends the notion of *local convergence* of graphs with bounded degrees of Benjamini and Schramm [3];
- the full set FO of all first-order formulas.

For a fragment X of first-order logic like those considered above, we have the following general analytic representation theorem, which can be seen as an extension of the representation of left limits of dense graphs by infinite exchangeable graphs [1,6,8] and of local limits of graphs with bounded degrees by unimodular distributions [3]:

Theorem 1.1 ([11]) *Let S be the Stone dual of the Lindenbaum-Tarski algebra defined by X, and let Γ be the automorphism group of S (note that Γ naturally acts on S). For each formula $\phi \in X$, we denote by f_ϕ the indicator function of the clopen subset of S dual to ϕ.*

To each finite σ-structure $\mathbf{A}$ corresponds (injectively) a Γ-invariant prob-

ability measure $\mu_{\mathbf{A}}$ on S such that for every formula $\phi \in X$ it holds

$$\langle \phi, \mathbf{A} \rangle = \int_S f_\phi(T)\, \mathrm{d}\mu_{\mathbf{A}}(T),$$

For every sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ of σ-structures the sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is X-convergent if and only if the measures $\mu_{\mathbf{A}_n}$ converge weakly. Moreover, if the sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is X-convergent then the measures $\mu_{\mathbf{A}_n}$ converge to some Γ-invariant probability measure μ with the property that for every formula $\phi \in X$ it holds

$$\lim_{n\to\infty} \langle \phi, \mathbf{A}_n \rangle = \int_S f_\phi(T)\, \mathrm{d}\mu(T).$$

The key question in this area is to find a "nicer" description of limit objects themselves. Examples of such descriptions are analytic (graphons [10], hypergraphons [5], and permutons [7]) or structure-like (Borel structures and graphings [3]).

The notion of structural convergence relies both on a representation of the combinatorial object (leading to the choice of a specific signature) and on the choice of a fragment of first-order logic. On the other hand different representations and different fragments may lead to same (or closely related) convergence.

The present status is summarized by table 1.

Table 1
Known limit objects

signature	QF-limit	$\mathrm{FO}^{\mathrm{local}}$-limit	FO-limit
one unary relation	Borel marking	Borel marking	Borel marking
one unary function	Theorem 2.1	Theorem 2.2	Theorem 2.3
d involutions	graphing	graphing	graphing
one binary symmetric relation	graphon	partial results [a]	
one binary symmetric function	very partial results [b]	?	?

[a] modeling FO-limits (hence modeling $\mathrm{FO}^{\mathrm{local}}$-limits) for sequences of graphs $(G_n)_{n\in\mathbb{N}}$ such that for every $r \in \mathbb{N}$ there is $F(r)$ such that for every $n \in \mathbb{N}$ and every $v \in G_n$, the number of cycles in the r-neighborhood of v is at most $F(r)$ [12,14].

[b] Borel QF-limit for tree semi-lattice infimum function [4].

For the natural representation of graphs with maximum degree d (has a relational structure), the notion of $\mathrm{FO}^{\mathrm{local}}$-convergence meets the notion of local convergence of Benjamini and Schramm. In this context, the limit of a convergent sequence of bounded degree connected graphs can be represented by a graphing, which is a Borel graph on a standard probability space that satisfies the *intrinsic mass transport principle* [3], which states that for every measurable subsets of vertices A, B it holds

$$\int_A \deg_B(x)\,\mathrm{d}\mu(x) = \int_B \deg_A(y)\,\mathrm{d}\mu(y),$$

where $\deg_B(x)$ (resp. $\deg_A(x)$) denotes the degree in B (resp. in A) of vertex x, and μ is the (atomless) probability measure on X. It is not known whether every graphing is the local limit of a sequence of connected finite graphs. This has been conjectured by Aldous and Lyons [2], but appears to be very difficult to prove or disprove.

Alternatively, a graphing can be represented as a standard probability space and D measure preserving involutions $f_1, \ldots, f_D$, two distinct vertices x and y being adjacent iff there is some $1 \leq i \leq D$ with $f_i(x) = y$. Note that the assumption that every f_i is measure-preserving directly implies the intrinsic mass transport principle. But finite graphs with degree less than D can similarly be represented as a σ-structure, where $\sigma = \{f_1, \ldots, f_D\}$ only contain function symbols interpreted as involutions, two vertices u and v being adjacent if $u \neq v$ and there is $1 \leq i \leq D$ such that $f_i(u) = v$. In this setting, the alternative form of graphings are QF-limits of bounded degree graphs represented as purely functional σ-structures.

For relational structures with a single relation symbol is QF-convergence essentially the same as left-convergence [13]. It follows from the particular cases of graphs [10] and regular hypergraphs [5] that QF-convergence of relational structures is essentially understood.

The general structures (including function symbols) can be of course reduced to relational structures but this reduction is not compatible with QF-convergence. The introduction of function symbols has great expressive power and this motivates our study of convergence of the simplest purely functional structures: mappings.

An r-ary function can be used to encode an r-ary relation. For instance, let $f : V \times V \to V$ have the property that $\{f(x,y), f(y,x)\} = \{x,y\}$ (such mappings, or operations are called *quasi-trivial*). Quasi-trivial functions allow us to encode any graph (by $x \sim y$ iff $f(x,y) = y$) and QF-convergence of these mappings is equivalent to left-convergence of the encoded graphs. This shows that r-ary functions present an intermediate level between r-ary relations and $(r+1)$-relations.

2 Statements of results

Our main result is that for "mappings" and all three main fragments we can characterize the limit objects and thus particularly prove analog of the Aldous-Lyons conjecture.

2.1 QF-*limits of Mappings*

When one considers left-convergence of graphs that are structurally sparse (meaning that one cannot find large "random-like" parts in them), natural limit object to consider for QF-convergence are *Borel structures*, that is structures whose domain is a standard Borel space equipped with an atomless probability measure, and whose relations and functions are Borel.

In the statement of the next theorem, an element $x \in A$ is called a *cyclic element* of a mapping $f : A \to A$ if there exists $n \in \mathbb{N}$ such that $f^n(x) = x$.

Theorem 2.1 *The* QF*-limit of a* QF*-convergent sequence of mappings $f_n : A_n \to A_n$ (with A_n finite, and $\lim_{n\to\infty} |A_n| = \infty$) can be represented as a Borel mapping $f : A \to A$, where A is a standard Borel space equipped with an atomless probability measure ν. Moreover, the mapping f is such that for every measurable subset X of the set of all cyclic elements of f it holds $\nu(f^{-1}(X)) = \nu(X)$.*

Conversely, every Borel mapping $f : A \to A$, where A is a standard Borel space equipped with an atomless probability measure ν and such that for every measurable subset X of the set of all cyclic elements of f it holds $\nu(f^{-1}(X)) = \nu(X)$ is the QF*-limit of a* QF*-convergent sequence of mappings $f_n : A_n \to A_n$ with A_n finite and $\lim_{n\to\infty} |A_n| = \infty$.*

2.2 $\mathrm{FO}^{\mathrm{local}}$-*limits of Mappings*

In the case of $\mathrm{FO}^{\mathrm{local}}$-convergence, the notion of Borel structure is too weak to allow to properly extend the notion of Stone pairing to formulas that contain quantifiers. Indeed, for every considered first-order formula ϕ the set of tuples satisfying ϕ should be measurable in order to extend the Stone pairing to the limit object. This leads to the notion of *modeling*: a modeling $\mathbf{A}$ is a Borel structure whose domain A is a standard Borel space equipped with a probability measure ν, with the property that that every first-order definable subset of A^k is measurable (by the product measure $\nu^{\otimes k}$).

A modeling mapping f satisfies the *finitary mass transport principle* if, for every measurable subsets X, Y of A, such that $X \subseteq \mathrm{Im} f$ and $\sup_{x\in X} |f^{-1}(x) \cap Y| < \infty$. Then

$$\nu(f^{-1}(X) \cap Y) = \int_X |f^{-1}(x) \cap Y| \,\mathrm{d}\nu(y)$$

These definitions fit to the context of $\mathrm{FO}^{\mathrm{local}}$-limits of mappings:

Theorem 2.2 *The* $\mathrm{FO}^{\mathrm{local}}$*-limit of an* $\mathrm{FO}^{\mathrm{local}}$*-convergent sequence of mappings* $f_n : A_n \to A_n$ *(with* A_n *finite, and* $\lim_{n\to\infty} |A_n| = \infty$*) can be represented by a mapping modeling* $f : A \to A$ *with atomless associated probability measure, such that* f *satisfies the finitary mass transport principle.*

Conversely, every modeling mapping $f : A \to A$ *with atomless associated probability measure that satisfies the finitary mass transport principle is the* $\mathrm{FO}^{\mathrm{local}}$*-limit of an* $\mathrm{FO}^{\mathrm{local}}$*-convergent sequence of mappings* $f_n : A_n \to A_n$ *with* A_n *finite and* $\lim_{n\to\infty} |A_n| = \infty$*.*

2.3 FO*-limits of Mappings*

In the case of full FO-convergence, some more restriction exists on the limit objects. Specifically, every sentence which is true on the limit object should have a finite model. In other words, the complete theory of the limit object should have the finite model property. It appears that this is the only additional requirement for FO-limits of mappings.

Theorem 2.3 *The* FO*-limit of an* FO*-convergent sequence of mappings* $f_n : A_n \to A_n$ *(with* A_n *finite, and* $\lim_{n\to\infty} |A_n| = \infty$*) can be represented by a mapping modeling* $f : A \to A$ *with atomless associated probability measure, whose complete theory has the finite model property, and such that* f *satisfies the finitary mass transport principle*

Conversely, every modeling mapping $f : A \to A$ *with atomless associated probability measure, whose complete theory has the finite model property, and that satisfies the finitary mass transport principle is the* FO*-limit of an* FO*-convergent sequence of mappings* $f_n : A_n \to A_n$ *with* A_n *finite and* $\lim_{n\to\infty} |A_n| = \infty$*.*

3 Examples

Let us give some intuition on the requirements of the theorems.

Example 3.1 Consider the mapping $f : [0,1] \to [0,1]$, defined as follows:

$$f(x) = \begin{cases} 1 - x & \text{if } x \notin [2/5, 3/5] \\ 2x - 4/5 & \text{otherwise} \end{cases}$$

Then the mapping f is the QF-limit of the mappings $f_n : A_n \to A_n$ where $A_n = \{a_1, \dots, a_{2n}\} \cup \{b_1, \dots, b_{2n}\} \cup \{c_1, \dots, c_n\}$ and f_n is defined by $f_n(a_i) = b - i$, $f_n(b_i) = a_i$ (for $1 \leq i \leq 2n$) and $f_n(c_i) = a_i$ (for $1 \leq i \leq n$). However, f is not the local-limit of a sequence of mappings on finite sets, for otherwise we can consider approximations $g_n : V_n \to V_n$ of f with V_n finite (and g_n converges to f). Let $C_n = \{x \in V_n : g_n \circ g_n(x) \neq x\}$ and $D_n = g_n(C_n) = \{x \in V_n : \exists y \ \ (g_n(y) = x) \wedge (g_n(x) \neq y)\}$. Then (denoting h the function symbol in σ to distinguish it from its interpretations f and g_n) it holds

$$\begin{aligned} \lim_{n\to\infty} \frac{|C_n|}{|V_n|} &= \lim_{n\to\infty} \langle (h(h(x_1)) \neq x_1), g_n \rangle = \langle (h(h(x_1)) \neq x_1), f \rangle = 1/5 \\ \lim_{n\to\infty} \frac{|D_n|}{|V_n|} &= \langle (\exists y \ \ (h(y) = x_1) \wedge (h(x_1) \neq y)), f \rangle = 2/5 \end{aligned}$$

although for every integer n, we have $|D_n| \leq |C_n|$ as $D_n = g_n(C_n)$.

Example 3.2 Consider the mapping $f : [0,1] \to [0,1]$, which maps x to $x/2$. This mapping is the QF-limit of the mapping $f_n : \{0, \dots, n\} \to \{0, \dots, n\}$ which maps n to n and $0 \leq i < n$ to $(i+1) \bmod n$. However, f is not an $\mathrm{FO}^{\mathrm{local}}$-limit of finite mappings. Indeed, it is bijective but $\nu(f^{-1}([0,1/2]) \neq \nu([0,1/2])$.

Example 3.3 Let $0 < \alpha < 1/2$ be irrational and let consider the mapping $f : [0,1) \to [0,1)$ defined by

$$f(x) = \begin{cases} (x - \alpha) \bmod 1 & \text{if } (\exists n \in \mathbb{N})\ x = n\alpha \bmod 1 \\ 0 & \text{if } x = 0 \\ (x + \alpha) \bmod 1 & \text{otherwise} \end{cases}$$

Then f has no first-order approximation: the sentence

$$(\forall x\, \exists y\, f(y) = x) \wedge (\exists x, y\, (x \neq y) \wedge f(x) = f(y)),$$

which means that f is surjective but not injective, belongs to the complete theory of f, but this sentence has no finite model as every $g : V \to V$ with V finite that is surjective has to be injective as well. Hence the complete theory of f does not have the finite model property. However, f is clearly the local limit of the mapping $f_n : \{0, \ldots, n-1\} \to \{0, \ldots, n-1\}$ which maps x to $(x+1) \bmod n$.

4 Concluding Remarks

Let us finish this extended abstract by saying that although the statement of Theorems 2.2 and 2.3 are intuitive (as also indicated by the above examples) we have no simple proof of them in either direction. The proofs build upon experience gained in treating limits of colored bounded height trees [13] and of trees in general [12], and uses a combinatorial analysis of the Stone dual on the Lindenbaum-Tarski algebra, which may be of independent interest.

References

[1] Aldous, D.: Exchangeability and continuum limits of discrete random structures. In: H.B. Agency (ed.) Proceedings of the International Congress of Mathematicians, vol. I, pp. 141–153 (2010)

[2] Aldous, D., Lyons, R.: Processes on unimodular random networks. Electronic Journal of Probability **12**, 1454–1508 (2007)

[3] Benjamini, I., Schramm, O.: Recurrence of distributional limits of finite planar graphs. Electron. J. Probab. **6**(23), 13pp (2001)

[4] Charbit, P., Hosseini, L., Ossona de Mendez, P.: QF-limits of structures and the example of tree-semilattices. arXiv:1505.03037 [math.CO] (2015)

[5] Elek, G., Szegedy, B.: Limits of hypergraphs, removal and regularity lemmas. A non-standard approach. arXiv:0705.2179v1 [math.CO] (2007)

[6] Hoover, D.: Relations on probability spaces and arrays of random variables. Tech. rep., Institute for Advanced Study, Princeton, NJ (1979)

[7] Hoppen, C., Kohayakawa, Y., Moreira, C., Ráth, B., Sampaio, R.: Limits of permutation sequences. Journal of Combinatorial Theory, Series B **103**(1), 93 – 113 (2013). DOI http://dx.doi.org/10.1016/j.jctb.2012.09.003

[8] Kallenberg, O.: Probabilistic Symmetries and Invariance Principles. Probability and Its Applications. Springer (2005)

[9] Lovász, L.: Large Networks and Graph Limits, *Colloquium Publications*, vol. 60. American Mathematical Society (2012)

[10] Lovász, L., Szegedy, B.: Limits of dense graph sequences. J. Combin. Theory Ser. B **96**, 933–957 (2006)

[11] Nešetřil, J., Ossona de Mendez, P.: A model theory approach to structural limits. Commentationes Mathematicæ Universitatis Carolinæ **53**(4), 581–603 (2012)

[12] Nešetřil, J., Ossona de Mendez, P.: Modeling limits in hereditary classes: Reduction and application to trees. arXiv:1312.0441 [math.CO] (2013)

[13] Nešetřil, J., Ossona de Mendez, P.: A unified approach to structural limits (with application to the study of limits of graphs with bounded tree-depth). arXiv:1303.6471v2 [math.CO] (2013)

[14] Nešetřil, J., Ossona de Mendez, P.: Local limits and asymptotic connectivity (2015). In preparation