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Uniform Linear Embeddings of Spatial Random Graphs

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Abstract

In a random graph with a spatial embedding, the probability of linking to a particular vertex v decreases with distance, but the rate of decrease may depend on the particular vertex v, and on the direction in which the distance increases. In this article, we consider the question when the embedding can be chosen to be uniform, so the probability of a link between two vertices depends only on the distance between them. We give necessary and sufficient conditions for the existence of a uniform linear embedding (embedding into a one-dimensional space) for spatial random graphs where the link probability can attain only a finite number of values.

Keywords: spatial graph model, linear embedding, random graph, geometric graph.

1 Introduction

In the study of large, real-life networks such as on-line social networks and hyperlinked "big data" networks, biological networks and neural connection networks, link formation is often modelled as a stochastic process. The underlying assumption of the link formation process is that vertices have an *a priori* identity and relationship to other vertices, which informs the link formation. These identities and relationships can be captured through an embedding of the vertices in a metric space, in such a way that the distance between vertices in the space reflects the similarity or affinity between the identities of the vertices. Link formation is assumed to occur mainly between vertices that have similar identities, and thus are closer together in the metric space. We take as our point of departure a very general stochastic graph model that fits the broad concept of graphs stochastically derived from a spatial layout, along the same principles as described above. We refer to this model as a *spatial random graph*. In a spatial random graph, vertices are embedded in a metric space, and the *link probability* between two vertices depends on this embedding in such a way that vertices that are close together in the metric space are more likely to be linked.

The concept of a spatial random graph allows for the possibility that the link probability depends on the spatial position of the vertices, as well as their metric distance. Thus, in the graph we may have tightly linked clusters for two different reasons. On the one hand, such clusters may arise when vertices are situated in a region where the link probability is generally higher. On the other hand, clusters can still arise when the link probability function is *uniform*, in the sense that the probability of a link between two vertices depends only on their distance, and not on their location. In this case, tightly linked clusters can arise if the distribution of vertices in the metric space is inhomogeneous. A cluster in the graph then points to a corresponding tightly packed cluster in the metric space. The central question addressed in this paper is how to recognize spatial random graphs with a uniform link probability function.

Here we study a one-dimensional spatial model where the metric space is $[0,1]$. Let $\mathcal{W}_0$ be the class of symmetric measurable functions from $[0,1]^2$ to $[0,1]$, and let $w \in \mathcal{W}_0$. The *w-random graph*, $G(n,w)$ is the graph with vertex set V of n points chosen uniformly from the metric space $[0,1]$. Then two vertices $x, y \in V$ are linked with probability $w(x,y)$. The w-random graphs was first introduced in [3]. For the graph $G(n,w)$ to correspond to a notion of spatial random graph, w must be such that points closer together have a higher probability of being linked. This implies that for $x < y$ the value of $w(x,y)$ decreases whenever x decreases or y increases. We call $w \in \mathcal{W}_0$ a *diagonally increasing function* if it satisfies the above property.

The notion of diagonally increasing functions and our interpretation of spatial random graphs were first given at a previous work. See [1]. Such random models have several real life applications (see [2], and [4] for applications in social networks, and [5] for neuroscience). In [1], a graph parameter

Γ is given which aims to measure the similarity of a graph to an instance of a one-dimensional spatial random graph model. However, the parameter Γ fails to distinguish uniform spatial random graph models from the ones which are intrinsically nonuniform. This natural question, which is a generalization of the question regarding interval graphs versus unit interval graphs, is the motivation behind this work.

We now formulate our central question: "which functions w are in fact uniform in disguise?"

Definition 1.1 A diagonally increasing function $w \in \mathcal{W}_0$ has a *uniform linear embedding* if there exists a measurable injection $\pi : [0,1] \to \mathbb{R}$ and a decreasing function $f_{pr} : \mathbb{R}^{\geq 0} \to [0,1]$ such that for every $x, y \in [0,1]$, $w(x,y) = f_{pr}(|\pi(x) - \pi(y)|)$.

In this paper we only study the diagonally increasing functions w with finite range. We will present necessary and sufficient conditions for a diagonally increasing finite-valued $w \in \mathcal{W}_0$ to admit a uniform linear embedding. The strength of our approach is the fact that it suggests an algorithm to construct the embedding.

2 Uniform linear embedding of finite-valued functions

In this section we study the properties of finite-valued diagonally increasing functions $w \in \mathcal{W}_0$. We will state our main result which provides necessary and sufficient conditions for the existence of a uniform linear embedding for diagonally increasing functions $w \in \mathcal{W}_0$. It turns out that w has a very specific form. For each $y \in [0,1]$, $w(x,y)$, viewed as a function of x, is a step function which is increasing for $x \in [0,y]$, and decreasing for $x \in [y,1]$. This function is determined by the boundary points where the function changes values. This leads to the following definition.

Definition 2.1 Let $w \in \mathcal{W}_0$ be a diagonally increasing function with range$(w) = \{\alpha_1, \ldots, \alpha_N\}$, where $\alpha_1 > \alpha_2 > \ldots > \alpha_N$. For $1 \leq i \leq N$, the *upper boundary* r_i and the *lower boundary* ℓ_i are functions from $[0,1]$ to $[0,1]$ defined as follows. Fix $x \in [0,1]$. Then $\ell_i(x) = \inf\{y \in [0,1] : w(x,y) \geq \alpha_i\}$ and $r_i(x) = \sup\{y \in [0,1] : w(x,y) \geq \alpha_i\}$. Also, for $1 \leq i < N$, define $r_i^* = r_i|_{[0,\ell_i(1)]}$ and $\ell_i^* = \ell_i|_{[r_i(0),1]}$. The function r_i^* has domain $[0, \ell_i(1)]$ and range $[r_i(0), 1]$, and ℓ_i^* has domain $[r_i(0), 1]$ and range $[0, \ell_i(1)]$.

Since w is diagonally increasing, we have $w(x,y) \geq \alpha_i$ if $y \in (\ell_i(x), r_i(x))$. On the other hand, $w(x,y) < \alpha_i$ whenever $y \in [0, \ell_i(x)) \cup (r_i(x), 1]$. Thus, the boundaries almost completely define w. By modification of w on a set of measure zero, we can assume that $w(x, r_i(x)) = w(x, \ell_i(x)) = \alpha_i$. We will assume throughout without loss of generality that for $1 \le i \le N$, the functions $\ell_i, r_i : [0,1] \to [0,1]$ satisfy

$$w(x,y) \geq \alpha_i \text{ if and only if } \ell_i(x) \leq y \leq r_i(x). \tag{1}$$

Note also that $r_N(x) = 1$ for all x. Therefore, we usually only consider the boundary functions r_i, ℓ_i for $1 \le i < N$. Below, we state an adaptation of Definition 1.1 for finite-valued diagonally increasing functions.

Definition 2.2 Let $w \in \mathcal{W}_0$ be a diagonally increasing function of finite range, defined as in Equation (1). Then w has a uniform linear embedding if there exists a measurable injection $\pi : [0,1] \to \mathbb{R}$ and real numbers $0 < d_1 < d_2 < \ldots < d_{N-1}$ so that for all $(x,y) \in [0,1]^2$,

$$w(x,y) = \begin{cases} \alpha_1 \text{ if } |\pi(x) - \pi(y)| \leq d_1, \\ \alpha_i \text{ if } d_{i-1} < |\pi(x) - \pi(y)| \leq d_i \text{ and } 1 < i < N, \\ \alpha_N \text{ if } |\pi(x) - \pi(y)| > d_{N-1}. \end{cases} \tag{2}$$

We call $d_1, d_2, \ldots, d_{N-1}$ the parameters of the uniform linear embedding π.

As we can see in Definition 2.2 the value of $w(x,y)$ is only a function of the distance of $\pi(x)$ and $\pi(y)$, the images of x and y in $[0,1]$ under the uniform linear embedding π.

We say that a function w is *well-separated* if the boundaries r_i and ℓ_i are continuous and boundaries r_i and ℓ_i have positive distance from each other and from the diagonal. In this paper we consider well-separated functions w.

The domain and range of ℓ_i^* and r_i^*, and thus any composition of such functions, are (possible empty) closed intervals. We will refer to $f_1 \circ \ldots \circ f_k$ as a *legal composition*, if each f_i belongs to $\{r_j^*, \ell_j^* : \ j = 1, \ldots, N-1\}$ and $\mathrm{dom}(f_1 \circ \ldots \circ f_k) \neq \emptyset$. We define the *signature* of the legal composition $f_1 \circ \ldots \circ f_k$ to be the $(N-1)$-tuple $(m_1, \ldots, m_{N-1})$, where m_i is the number of occurrences of r_i^* minus the number of occurrences of ℓ_i^* therein. We use Greek letters such as $\phi, \psi, \ldots$ to denote legal compositions. We emphasize that a legal composition is a function, which we denote by legal function, presented in a particular manner as a composition of boundary functions. Note that two legal compositions may be identical as functions, but have different signatures, due to difference in their presentations. Legal compositions provide us with appropriate "steps" to define constrained points.

Definition 2.3 Let $w \in \mathcal{W}_0$ be a diagonally increasing function with finite range. Keep notations as in Definition 2.1, and define

$$\mathcal{P} = \{\phi(0) : \ \phi \text{ is a legal composition with } 0 \in \mathrm{dom}(\phi)\},$$
$$\mathcal{Q} = \{\psi(1) : \ \psi \text{ is a legal composition with } 1 \in \mathrm{dom}(\psi)\}.$$

We refer to $\mathcal{P} \cup \mathcal{Q}$ as the set of *constrained points* of w.

Note that the sets $\mathcal{P}$ and $\mathcal{Q}$ are either disjoint or identical. In this paper for simplicity we consider the case $\mathcal{P} \cap \mathcal{Q} = \emptyset$. We will study the case $\mathcal{P} = \mathcal{Q}$ in future work.

Definition 2.4 Assume a positive integer N and real numbers $d_{N-1} > \ldots > d_1 > 0$ are given. The *displacement* of a legal composition ϕ, denoted by $\delta(\phi)$, is defined as

$$\delta(\phi) = d_1 m_1 + \ldots + d_{N-1} m_{N-1},$$

where $(m_1, m_2, \ldots, m_{N-1})$ is the signature of ϕ.

Now we are ready to present necessary and sufficient conditions for a diagonally increasing finite-valued function w to admit a uniform linear embedding.

Theorem 2.5 *Let $w \in \mathcal{W}_0$ be a well-separated finite-valued diagonally increasing function assuming values $\alpha_1 > \alpha_2 > \ldots > \alpha_N$. Let $\mathcal{P}$ and $\mathcal{Q}$ be as in Definition 2.3 and $\mathcal{P} \cap \mathcal{Q} = \emptyset$. The function w has a uniform linear embedding if and only if the following conditions hold:*

(1) If ϕ is a legal composition with $\phi(x) = x$ for some $x \in \mathrm{dom}(\phi)$, then ϕ is the identity function on its domain.

(2) There exist real numbers $0 < d_1 < \ldots < d_{N-1}$ such that

(2a) The displacement δ as defined in Definition 2.4 is increasing on $\mathcal{P}$, in the sense that, for all $x, y \in \mathcal{P}$, and legal compositions ϕ, ψ so that $x = \phi(0)$ and $y = \psi(0)$, we have that, if $x < y$ then $\delta(\phi) < \delta(\psi)$.

(2b) The values $0 < d_1 < \ldots < d_{N-1}$ satisfy the following condition. Fix $1 \leq i \leq N-1$. Let $y \in \mathcal{Q}$ be such that $y < r_i^*(0)$, and x be any element of $\mathcal{P}$. Let ϕ and ψ be legal compositions so that $y = \phi(1)$ and $x = \psi(0)$. Then there exists $a \in \mathbb{R}^{\geq 0}$ such that $\delta(\psi) < a < d_i - \delta(\phi)$.

3 Necessary and sufficient conditions

This section is devoted to a brief overview of the proof of Theorem 2.5. Assuming w admits a uniform linear embedding π, we will state some important properties of π. These properties are the main components of the proof for the necessity part.

If the boundaries of w in Definition 2.1 are such that $\ell_1^*(1) < r_1^*(0)$ then it is easy to construct an increasing uniform linear embedding π for w. Now suppose that $\ell_1^*(1) > r_1^*(0)$ and w admits a uniform linear embedding π. In this case the following lemma shows that π is strictly monotone.

Lemma 3.1 *Let $w \in \mathcal{W}_0$ be a diagonally increasing function with finite range, defined as in Equation (1). Assume that w is well-separated, and admits a uniform linear embedding $\pi : [0,1] \to \mathbb{R}$ with parameters $0 < d_1 < d_2 < \ldots < d_{N-1}$ as in Definition 2.2. Let $\ell_1^*(1) > r_1^*(0)$. Then π is strictly monotone. In particular, π is continuous on all except countably many points in $[0,1]$.*

We can assume without loss of generality that $\pi(0) < \pi(r_i^*(0))$ and hence π is strictly increasing.

Let $\pi^+(x)$ and $\pi^-(x)$ determine the right and the left limits of π at the point $x \in (0,1)$. The following proposition shows that π is largely determined by the displacement of the legal compositions that generate points of $\mathcal{P}$.

Proposition 3.2 *Let $w \in \mathcal{W}_0$ be a well-separated, diagonally increasing finite-valued function. Assume that w admits a uniform linear embedding $\pi : [0,1] \to \mathbb{R}$ with parameters $0 < d_1 < d_2 < \ldots < d_{N-1}$ as in Definition 2.2. Let ϕ be a legal composition with non-empty domain $[p,q]$, where $p \in \mathcal{P}$ and $q \in \mathcal{Q}$. Then*

(i) $\pi^+(\phi(x)) - \pi^+(x) = \delta(\phi)$ *for every* $x \in [p,q)$.

(ii) $\pi^-(\phi(x)) - \pi^-(x) = \delta(\phi)$ *for every* $x \in (p,q]$.

(iii) If $\delta(\phi) = 0$, *then* ϕ *is the identity function on its domain.*

The necessity of the conditions of Theorem 2.5 follows from Lemma 3.1 and Proposition 3.2. To prove the sufficiency we will construct a uniform linear embedding π. There are some issues that we need to take into account in the construction of π. For example the definition of π on intervals $[x,y]$ and $[r^*(x), r^*(y)]$ are closely related. To take care of such relations we will define an equivalence relation among some intervals on $[0,1]$.

Lemma 3.3 *Let w be a diagonally increasing $\{\alpha_1, \ldots, \alpha_N\}$-valued function as define by Equation (1), and let $\mathcal{P}$ and $\mathcal{Q}$ be as in definition 2.3. Assume there exist real numbers $d_1 < \ldots < d_{N-1}$ such that conditions (1), (2a), and (2a) of Theorem 2.5 hold. Let $\overline{\mathcal{P} \cup \mathcal{Q}}$ denote the closure of $\mathcal{P} \cup \mathcal{Q}$ in the usual topology of $[0,1]$. For a countable index set I and pairwise disjoint open intervals I_i, we have $[0,1] \setminus (\overline{\mathcal{P} \cup \mathcal{Q}}) = \cup_{i \in I} I_i$. We say $i \sim i'$ if there exists a legal composition ϕ such that $I_i = \phi(I_{i'})$. Then the relation $\sim$ is an equivalence relation.*

We now define a strictly increasing uniform linear embedding π in stages. Here we avoid the details and just give a general insight into the process of constructing π.

(1) Define π on $\mathcal{P} \cup \mathcal{Q}$.
- Let $x \in \mathcal{P}$ and $\phi(0) = x$, where ϕ is a legal composition. Then $\pi(0) = 0$ and $\pi(x) = \delta(\phi)$.
- We pick $\pi(1)$ from an interval obtained from conditions of Theorem 2.5, and for $y \in \mathcal{Q}$ we define $\pi(y) = \pi(1) + \delta(\psi)$ where ψ is a legal composition with $\psi(1) = y$.

(2) Define π on $\overline{\mathcal{P} \cup \mathcal{Q}}$. We define $\pi(x)$, for x in the completion of $P \cup Q$, using sequences converging to x from the left (if possible).

(3) Extend π to $[0,1] \setminus \overline{\mathcal{P} \cup \mathcal{Q}}$. We use the equivalence relation of Lemma 3.3 to extend π to $[0,1] \setminus \overline{\mathcal{P} \cup \mathcal{Q}}$ as follow. First note that $[0,1] \setminus \overline{\mathcal{P} \cup \mathcal{Q}} = \cup_{i \in I} I_i$ for a countable index set I. For each equivalence class $[I_i]$, proceed as follows. First, pick a representative I_i for $[I_i]$, and define π on I_i. Then, for every $I_j \in [I_i]$, let ϕ is a legal composition with $\phi(I_j) = I_i$. We define $\pi(x)$ according to the definition of π on I_i, *i.e.*, for $x \in I_j$

$$\pi(x) = \pi(\phi(x)) - \delta(\phi).$$

This gives us a strictly increasing uniform linear embedding on $[0,1]$.

References

[1] H. Chuangpishit, M. Ghandehari, M. Hurshman, J. Janssen, and N. Kalyaniwalla, Linear Embeddings of Graphs and Graph Limits, *J. Combin. Theory Ser. B* **113** (2015) 162–184.

[2] P. D. Hoff, A. E. Raftery, and M. S. Handcock. Latent space approaches to social network analysis. *Journal of the American Statistical Association* **97(460)** (2002) 1090–1098.

[3] L. Lovász and B. Szegedy, Limits of dense graph sequences *J. Combin. Theory Ser. B* **96(6)** (2006) 933–957.

[4] M. McPherson, Blau space primer: prolegomenon to an ecology of affiliation, *Industrial and Corporate Change* **13(1)** (2004) 263–280.

[5] V. Pernice, B. Staude, S. Cardanobile, and S. Rotter, How Structure Determines Correlations in Neuronal Networks *PLoS Comput Biol* **7(5)** (2011).