Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in DISCRETE MATHEMATICS

www.elsevier.com/locate/endm

Limits of some combinatorial problems

Endre Csóka*

Abstract

We show some new examples how can limit theory help understanding combinatorial structures. We introduce two limit problems of Alpern's Caching Game, which are good approximations of the original game when some parameters tend to infinity. With the use of these limit problems, we show some surprising results which radically changes our expectations about the structure of the optimal solution, e.g. this disproves the Kikuta-Ruckle Conjecture for Caching Games. For another example, we generalize the Manickam–Miklós–Singhi Conjecture, using limit theory.

1 Alpern's Caching Game

A summary about the results and connections of Alpern's Caching Game can be found in [1].

Definition 1. *There are n holes and k nuts. The hider places (digs) k nuts so that the sum of the depths of the deepest nuts in the different holes (0 if no nut in the hole) must be at most 1. The searcher cannot observe it, but he can dig at most h depth in total in a dynamic way, observing whenever a nut is found during the digging. The searcher wins if he finds at least j out of the k nuts.*

Definition 2. *In Alpern's Caching Game, we say that a placement of the nuts is* ***extremal*** *if this requires total digging length 1, and any less is not enough. A hiding strategy is called extremal if it is supported on extremal placements. The extremal version of the game means that the hider must use an extremal strategy.*

Question 3. *Does the hider always have an extremal strategy which is optimal? Or (equivalently) are the values of the game and its extremal version the same?*

We note that if the Kikuta-Ruckle Conjecture for Caching Games is true, then this implies positive answer to Question 3, by induction on k.

Now we define the limit of the game when the number of nuts to hide k and to find j are fixed, but the number of holes and the digging time $n, h \to \infty$.

Definition 4. *The* ***limit game*** *with parameter λ is defined as follows. The hider chooses k values (depths) $y_1, y_2, ...y_k \in [0,1]$ where $\sum y_i \le 1$ in the original, $\sum y_i = 1$ in the extremal version. Then for k independent uniform random numbers $x_1, x_2, ...x_k \in [0, \lambda]$, the nuts are placed at (x_i, y_i). The searcher observes nothing. Now the searcher should define a function $f_t(x) : [0,1] \times [0, \lambda] \to [0,1]$ which is monotone increasing in both parameters and $\int f_1(x)\,\mathrm{d}x \le 1$. Then we evaluate f meaning that the searcher gets to know the smallest t^* that some $f_{t^*}(x_i) \ge y_i$. If there is no such a nut even for f_1, then hider wins. Otherwise the nut at (x_i, y_i) is found by the searcher, he gets to know its position, and we remove this nut. Then the searcher can change his function in the parameter interval $t \in (t^*, 1]$, and we re-evaluate f. The searcher wins if he finds at least j nuts in total.*

*Institute of Mathematics and DIMAP, University of Warwick, Coventry, United Kingdom.

Theorem 5. *Consider the game with parameters k, j, n, h and consider the limit game with $k, j, \lambda = n/h$. The ratio of the values of these two games $\in \left[\left(\frac{h-j}{h}\right)^j, 1\right]$.*

Sketch of proof. On one hand, any strategy of the searcher in the finite game can be applied in the limit game by choosing f_t in the interval $\left[\frac{i-1}{n}, \frac{i}{n}\right)$ as the depth of the ith deepest hole after t total digging. This way the searcher can get at least the same score as in the finite game.

On the other hand, a strategy of the searcher in the limit game can be applied in the finite game by having depth $f\left(\frac{i-1}{h-j}\right)$ in the ith hole, except that if a nut is found in a hole, then we dig that hole until depth 1. (The holes are randomly numbered by the searcher.) This way, the searcher can get at least $\left(\frac{h-j}{h}\right)^j$ times the score of the limit game. □

Definition 6. *The **double limit game** is defined as the limit game with $\lambda \to \infty$, as follows. The hider chooses k values $y_1, y_2, ...y_k \in [0, 1]$ where $\sum y_i \leq 1$ in the original, $\sum y_i = 1$ in the extremal version. Then with all possible vector of positive real numbers $x_1, x_2, ...x_k \in \mathbb{R}^+$, the nuts are placed at (x_i, y_i). The searcher observes nothing. Now the searcher should define a strategy of the limit game with $\lambda = \infty$. The score of the searcher is the j-dimensional measure of the vectors $x_1, x_2, ...x_k$ for which he wins. This is what the searcher aims to maximize and the hider aims to minimize, in expectation.*

Theorem 7. *Fix k and j, and consider a sequence of pairs (n_i, h_i) so that $h_i \to \infty$ and $n_i/h_i \to \infty$. Then the values of the games with parameters k, j, n_i, h_i, multiplied by $\left(\frac{n_i}{h_i}\right)^j$, tend to the value of the double limit game.*

Sketch of proof only for $k = j = 2$. The strategy of the searcher in a limit game can be applied in the double limit game, providing the same score. This proves one direction.

The other direction is a bit more technical. The optimal strategy of the searcher in the double limit game can be applied in the limit game with a large parameter λ, simply by restricting f to $[0, 1] \times \lambda$. In the part which is cut, the function value was at most $\frac{1}{\lambda}$, therefore, this strategy provides the same score unless if the depth of a nut is very low. However, there is a strategy which is very efficient in these cases: $f_1(x) = 1$ if $x < \frac{1}{2}$, and $f_1(x) = \frac{1}{\lambda}$ if $x \in \left[\frac{1}{2}, \frac{\lambda+1}{2}\right)$, and 0 otherwise. If we use this strategy instead with an appropriate very low probability, then for all possible hiding strategy, this mixed searching strategy will be at least almost as good as the original strategy in the double limit game. □

Theorem 8. *If $k = j = 2$, then the values of the double limit game and its extremal versions have different values. The same is true for the limit game with λ large enough.*

Notice that this theorem provides infinitely many counterexamples to Question 3.

Sketch of proof. Assume by contradiction that the two values are the same. Consider the optimal strategy of the searcher in the original game. This must be an optimal strategy in the extremal game, as well.

Consider any optimal strategy of the hider in the extremal game. This can be identified with the probability measure μ of the depth of a random nut.

Lemma 9. *If the searcher finds a nut at depth y, then he should change f_1 so as to maximize the interval where $f_1(x) = 1 - y$, except for a 0-measure set with respect to μ.*

In the extremal game, a mixture of the following two pure strategies of the searcher provides him score $\sqrt{2} + 1$. (The optimum score is ≈ 2.7.)

- $f_t^{(1)}(x) = 1$ if $x < t$, otherwise 0.
- $f_t^{(2)}(x) = \frac{1}{2}$ if $x < 2t$, otherwise 0.

Lemma 10. *With the optimal strategy of the searcher in the extremal game, the values of f_1 must be almost everywhere 0 or inside the closure $C = cl\big(supp(\mu)\big)$ of the support of the depths in an optimal strategy of the hider. More precisely, the event that this does not hold should have probability 0.*

Proof. Otherwise we could rounded down f to $C \cup \{0\}$ for the entire strategy of the searcher – with the appropriate reparametrization – which would not change the score of the searcher. If this rounding provides a jump in the integral, then we smooth it by increasing f from left to right while t increases. We will have some residue at the end because of rounding down, this could be used to increase the score. $\square$

Without loss of generality, we can assume that $\int f_t(x)\ \mathrm{d}x = t$.

Lemma 11. *If $y \notin C \cup \{0\}$ then for any $t \in [0,1]$, for all $x \in \mathbb{R}$ except a 0-measure set, $f_t(x) \neq y$.*

Proof. If it does not hold, then we could easily improve the strategy of the searcher. $\square$

Lemma 12. $C = [0,1]$.

Proof. Assume by contradiction that an interval $(a,b) \subset [0,1] - C$, but $a, b \in C$. Using the previous lemmas and the strict convexity of the function $x \to 1/x$, we get that $\big(\frac{a+b}{2}, 1 - \frac{a+b}{2}\big)$ provides lower score for the searcher than the average score by $(a, 1-a)$ and $(b, 1-b)$, which contradicts with the optimality of the strategy of the hider. For $a = 0$, the hiding strategy $\big(\frac{a+b}{2}, 1 - \frac{a+b}{2}\big)$ is better than $(b, 1-b)$. $\square$

Consider now the optimal strategy of the searcher in the non-extremal game. We know that if he finds a nut, then he follows the strategy according to Lemma 9. We also know that this must be an optimal strategy of the extremal game, as well, because the values of the two games are (assumed to be) the same. Now consider the case when the hider chooses depths $y_1 = \frac{1}{4} - \varepsilon$ and $y_2 = \frac{1}{4}$, for some sufficiently small ε. Consider a strategy of the searcher. Let

$$t^* = \sup_{t \in [0,1]} \left\{ f_t\Big(\frac{4}{3}\Big) < \frac{1}{4} \right\},$$

$$s = \sup_{x \in \mathbb{R}} \left\{ \sup_{t \in [0,1]} \left\{ f_t(x) < \frac{1}{4} \right\} = t^* \right\}.$$

It is easy to check that the searcher cannot win if $y_1 > s$ or $y_2 > 2 - \frac{s}{2}$ (if ε is small enough). Therefore, the score of the searcher is at most $s(2 - \frac{s}{2}) \le 2 < \sqrt{2} + 1$, which is a contradiction. $\square$

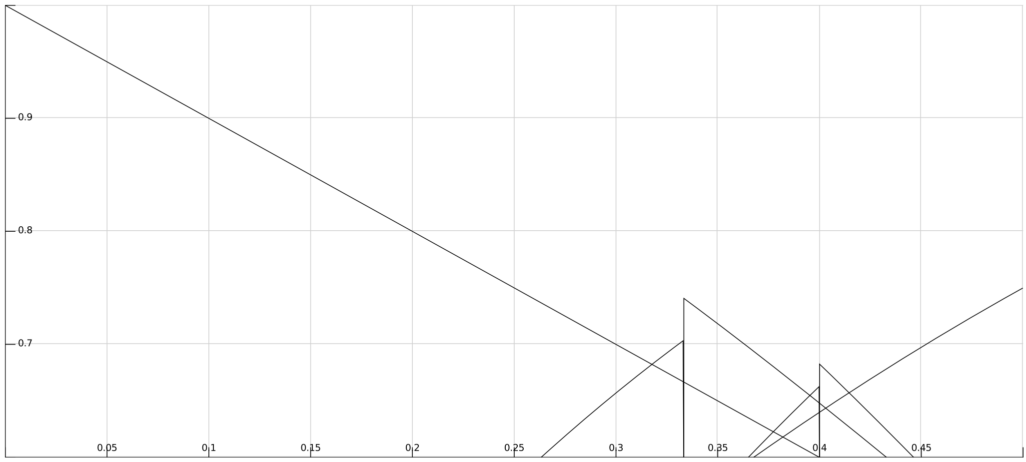
To sum up, this argument implies a surprising negative answer to Question 3 (and disproves the Kikuta-Ruckle Conjecture for Caching Games).

We note that there are a number of further limit games, e.g. if $j_i \to \infty$ and j_i/k_i is convergent. These limit games might also be interesting.

2 Manickam–Miklós–Singhi Conjecture

These limit theory techniques can be applied in other combinatorial problems, for example, we can generalize the Manickam–Miklós–Singhi Conjecture as follows.

Conjecture 13 (Manickam–Miklós–Singhi)**.** *If $n, k \in \mathbb{N}$, $4k \le n$, $a_i \in \mathbb{R}$, $a_1 + a_2 + ... + a_n = 0$ and $\{i_1, i_2, ..., i_k\} \subset \{1, 2, ..., n\}$ is a uniform random subset with fix cardinality k, then* $\Pr\big(a_{i_1} + a_{i_2} + ... + a_{i_k} > 0\big) \le \frac{n-k}{n}$ *with equality if and only if the sum of each k elements excluding one specific (smallest) element is positive (e.g. $1-n, 1, 1, 1, ..., 1$).*



If p is between 0 and 0.317... (or exactly 1/3), then -1 is the best (and the other coefficients are 0), between 0.317... and 1/3 the sequence $1, 1, 1$ is the best, between 1/3 and 0.395... (and at 0.4) the sequence $-1, -1, -1$, between 0.395... and 0.4 the sequence $1, 1, 1, 1, 1$, between 0.4 and 0.414... the sequence $-1, -1, -1, -1, -1$, and between 0.414... and 0.5 the sequence $1, 1$.

The Manickam–Miklós–Singhi Conjecture is introduced in 1987 in [2], and it has recently received a lot of attention, especially because of its connection to the Erdős matching conjecture. Now we define the limit problem as follows.

Problem 14. *For a fix $p \in [0, 1]$, we are looking for a countable sequence $a_1, a_2, ...$ of real numbers with $\sum a_i^2 < \infty$ and real number d which maximizes $\Pr\left(\sum a_i x_i + d x_0 > p \sum a_i\right)$, where $x_0, x_1, x_2, ...$ are independent, $x_1, x_2, ...$ are indicator variables with probability p, and x_0 is a variable with standard normal distribution.*

Conjecture 15. *The optimal strategy of the limit game has the form $a_1 = a_2 = ... = a_q$ where $q \in \{1, 2, 3, 5\}$, and all other coefficients are 0. This is the only optimal strategy up to equivalence.*

Two sequences are equivalent if the two events that the sum is positive are the same up to a permutation on the variables $x_1, x_2, ...$.

Figure 1 shows the probabilities by the different sequences, as a function of p. Now we are ready to state the corresponding conjecture in the original problem.

Conjecture 16. *The optimal strategy of the original finite game has the form $a_1 = a_2 = ... = a_q$ where $q \in \{1, 2, 3, 5\}$, and $a_{q+1} = a_{q+2} = ... = a_n = -\frac{q}{n-q} a_1$. This is the only optimal strategy up to equivalence.*

Problem 17. *There are $n, k \in \mathbb{N}$, and a vector space V over $\mathbb{R}$ and $T \subset V$ is a convex set. We want to find vectors $a_1, a_2, ..., a_n$ with $a_1 + a_2 + ... + a_n = 0$ which maximizes $\Pr\big(a_{i_1} + a_{i_2} + ... + a_{i_k} \in T\big)$, where $\{i_1, i_2, ..., i_k\} \subset \{1, 2, ..., n\}$ is a uniform random k-element subset.*

Conjecture 18. *The optimal strategy of this game has the form $a_1 = a_2 = ... = a_q$ and $a_{q+1} = a_{q+2} = ... = a_n = -\frac{q}{n-q} a_1$. q is bounded (approximately at most 20). This is the only optimal strategy up to equivalence.*

References

[1] Steve Alpern, Robbert Fokkink, G Leszek, Roy Lindelauf, VS Subrahmanian, et al. *Search theory: a game theoretic perspective.* Springer Science & Business Media, 2013.

[2] N Manickam and D Miklós. On the number of non-negative partial sums of a non-negative sum. In *Colloq. Math. Soc. János Bolyai*, volume 52, pages 385–392, 1987.